Vacuum birefringence and real-photon decay in strong magnetic fields

as a building block of phenomenology

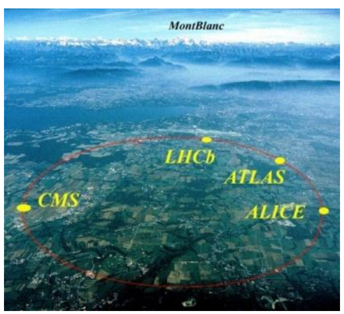
Koichi Hattori (RIKEN)

In collaboration with K. Itakura (KEK).

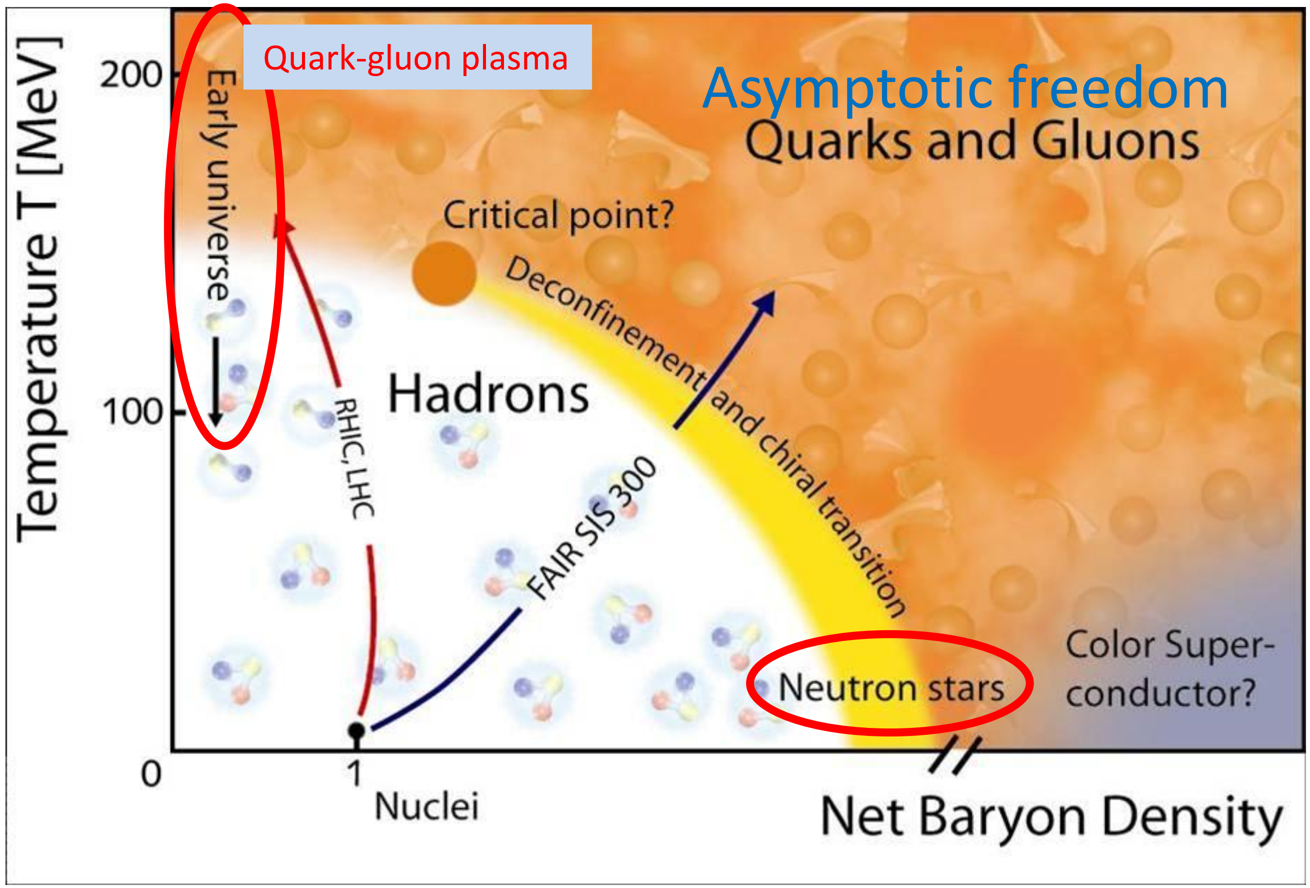
(I) KH, K. Itakura, Annals Phys. 330 (2013) 23-54
(II) KH, K. Itakura, Annals Phys. 334 (2013) 58-82

Phase diagram of QCD matter

RHIC@BNL

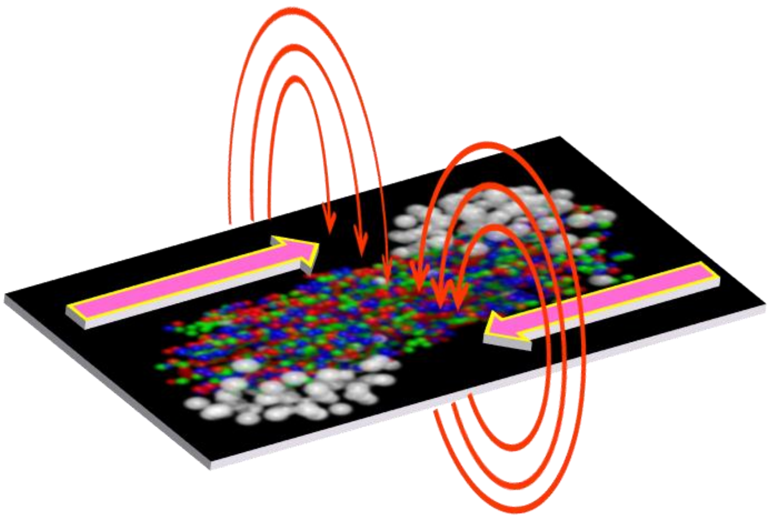


LHC@CERN



Extremely strong magnetic fields

UrHIC



Lienard-Wiechert potential

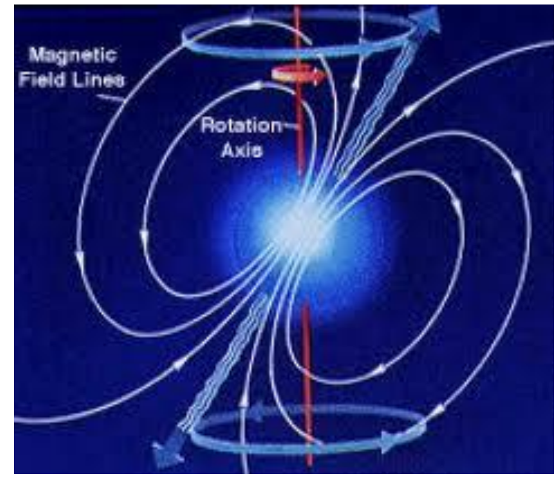
$$eB = \frac{\alpha_{\rm EM} Z \sinh(Y)\ (\boldsymbol{x}_\perp \times \boldsymbol{e}_z)}{[\ |\boldsymbol{x}_\perp|^2 + \tau^2 \sinh^2(Y-\eta)\ ]^{3/2}}$$

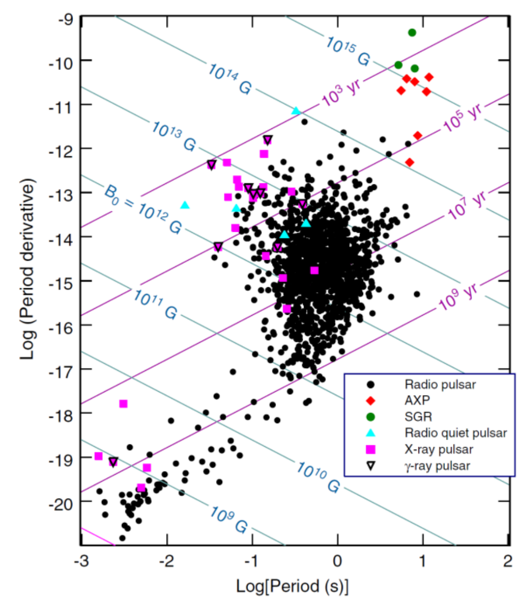
$Y = \tanh^{-1} v_{\rm N}$: beam rapidity $(v_N > 0.9999c)$

$\boldsymbol{x}_\perp$: distance from the beam in the transverse plane

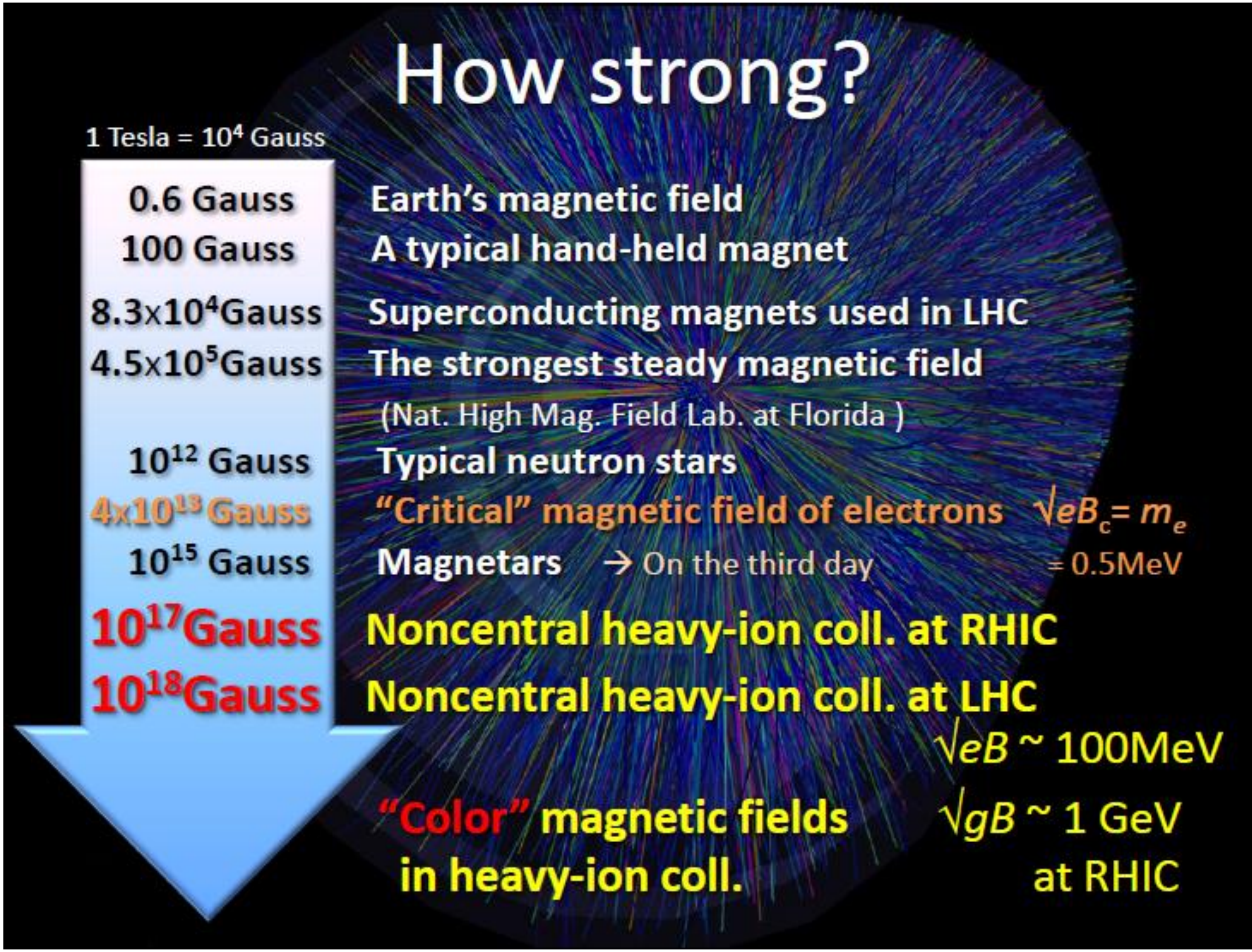
Z = 79(Au), 82(Pb)

NS/Magnetar

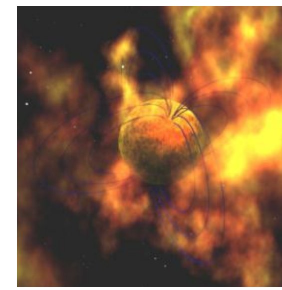




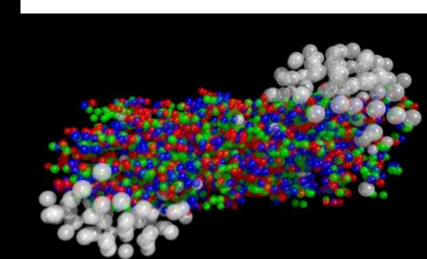
Strong magnetic fields in nature and laboratories



Magnet in Lab.



Magnetar



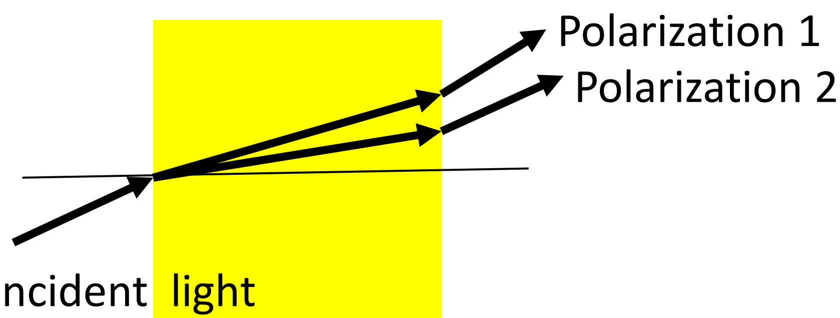
Heavy ion collisions

What is "Birefringence" ?

Two polarization modes of a propagating photon
have different refractive indices.

Doubled image due to a ray-splitting in birefringent substances

"Calcite" (方解石)



How about the vacuum with external magnetic fields ?

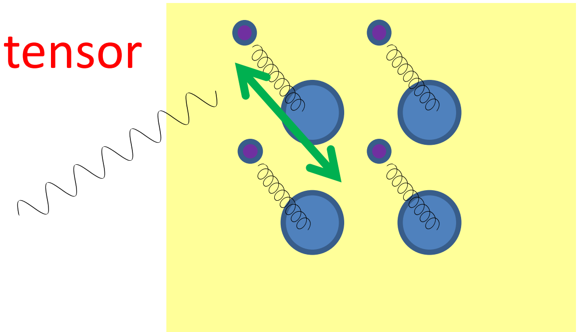
+ ~~Lorentz~~ & Gauge symmetries → n ≠ 1 in general

+ Oriented response of the Dirac sea → Vacuum birefringence

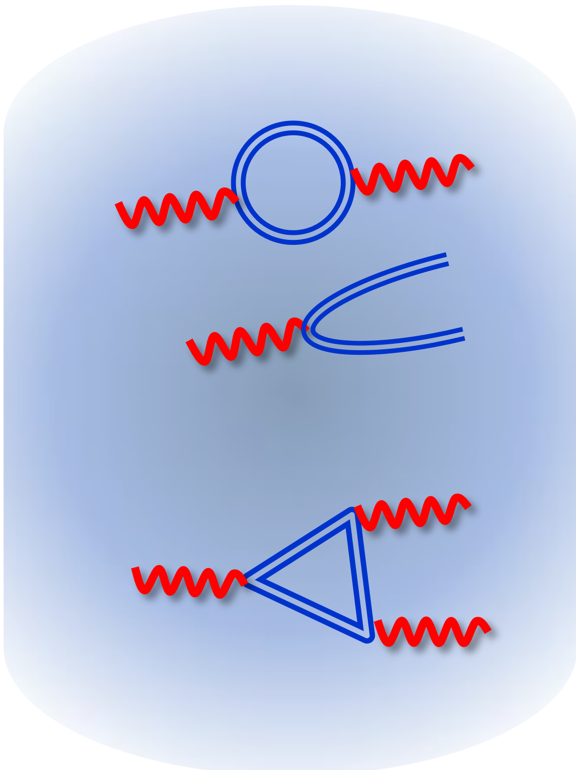
Modifications of photon propagations in strong B-fields
- Old but unsolved problems

Modified Maxwell eq. : $\left\{ q^2\eta^{\mu\nu} - q^\mu q^\nu - \Pi^{\mu\nu}_{\rm ex}(q^2) \right\} A_\nu(q) = 0$

Polarization tensor



Quantum effects in magnetic fields



Refraction of photons in the vacuum with B-fields without medium effects

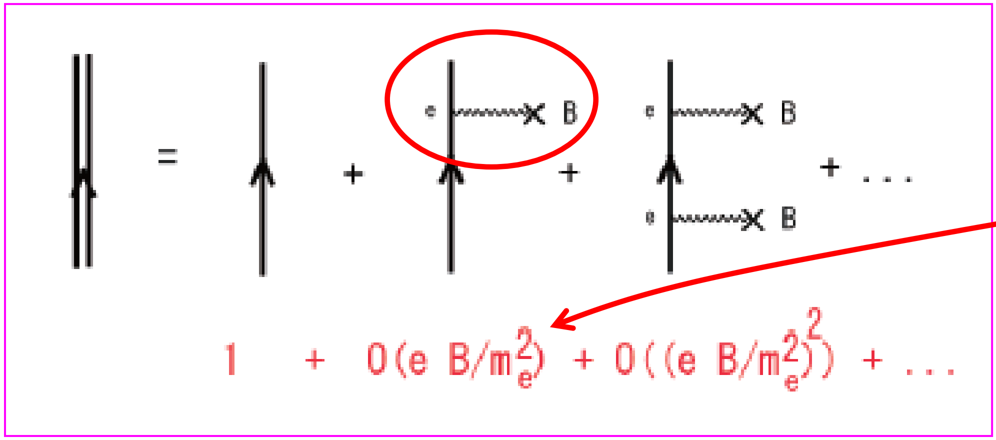
Real photon decay

Should be suppressed in the ordinary perturbation.

Photon splitting

Why strong B-fields?
- Break-down of naïve perturbation

Dressed fermion propagator in Furry's picture



Critical field strength
$B_c = m_e^2 / e$

In heavy ion collisions,
$B/B_c \sim O(10^4) >> 1$

Naïve perturbation breaks down when $B > B_c$

→ Need to take into account all-order diagrams

Resummation w.r.t. external legs by "proper-time method" Schwinger

$$G(p|A) = \frac{i\,(\not p - e\not A + m)}{(\not p - e\not A)^2 - m^2 + i\epsilon}$$

$$= i\,(\not p - e\not A + m) \times \frac{1}{i}\int_0^\infty d\tau\; e^{i\tau\{(\not p - e\not A)^2 - (m^2 - i\varepsilon)\}}$$

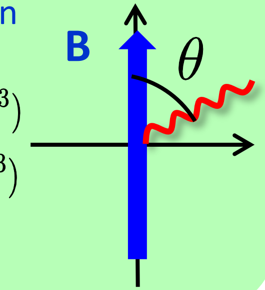
τ : proper-time

Nonlinear to strong external fields

Photon propagation in a constant external magnetic field

θ: angle btw B-field and photon propagation

$$q^\mu = (q^0, q_\perp, 0, q^3)$$
$$q_\parallel^\mu = (q^0, 0, 0, , q^3)$$
$$q_\perp^\mu = (0, q_\perp, 0, 0)$$



Lorentz and gauge symmetries lead to a tensor structure,

$$\Pi^{\mu\nu}_{\rm ex}(q^2) = -\left\{\chi_0 P_0^{\mu\nu} + \chi_1 P_1^{\mu\nu} + \chi_2 P_2^{\mu\nu}\right\}$$

$$P_0^{\mu\nu} = q^2\eta^{\mu\nu} - q^\mu q^\nu$$

$$P_1^{\mu\nu} = q_\parallel^2\eta_\parallel^{\mu\nu} - q_\parallel^\mu q_\parallel^\nu$$
$$P_2^{\mu\nu} = q_\perp^2\eta_\perp^{\mu\nu} - q_\perp^\mu q_\perp^\nu$$

$$\chi_i(r_\parallel^2, r_\perp^2, B_{\rm r}) = \frac{\alpha B_{\rm r}}{4\pi}\int_{-1}^{1} d\beta \int_0^\infty d\tau\ \frac{\Gamma_i(\tau,\beta)}{\sin(\tau)}\ e^{-i(\phi_\parallel + \phi_\perp)\tau}$$

Integrands having strong oscillations

$$\begin{cases} \Gamma_0(\tau,\beta) = \cos(\beta\tau) - \beta\sin(\beta\tau)\cot(\tau) \\ \Gamma_1(\tau,\beta) = (1-\beta^2)\cos(\tau) - \Gamma_0(\tau,\beta) \\ \Gamma_2(\tau,\beta) = 2\frac{\cos(\beta\tau)-\cos(\tau)}{\sin^2(\tau)} - \Gamma_0(\tau,\beta) \quad . \end{cases}$$

Schwinger, Adler, Shabad, Urrutia, Tsai and Eber, Dittrich and Gies

$$\phi_\parallel(r_\parallel^2, B_{\rm r}) = \frac{1}{B_{\rm r}}\left\{1 - (1-\beta^2)\ r_\parallel^2\right\}$$

$$\phi_\perp(r_\parallel^2, B_{\rm r}) = -\frac{2r_\perp^2}{B_{\rm r}}\cdot\frac{\cos(\beta\tau)-\cos(\tau)}{\sin(\tau)}$$

Exponentiated trig-functions generate strongly oscillating behavior with arbitrarily high frequency.

Analytic results of integrals without any approximation

KH, K. Itakura (I)

Every term results in either of three simple integrals.

$$\int_{-1}^{1} d\beta \int_{0}^{\infty} d\tau\, \beta^k\, e^{-i\,\Phi(\beta)\tau} \quad k = 0,\ 1,\ 2$$

$$\Phi(\beta) = \frac{1}{B_{\mathrm{r}}}\left\{1 - (1-\beta^2)\, r_\parallel^2\right\} + 2\ell - n\beta + n$$

Decomposition into a double infinite sum

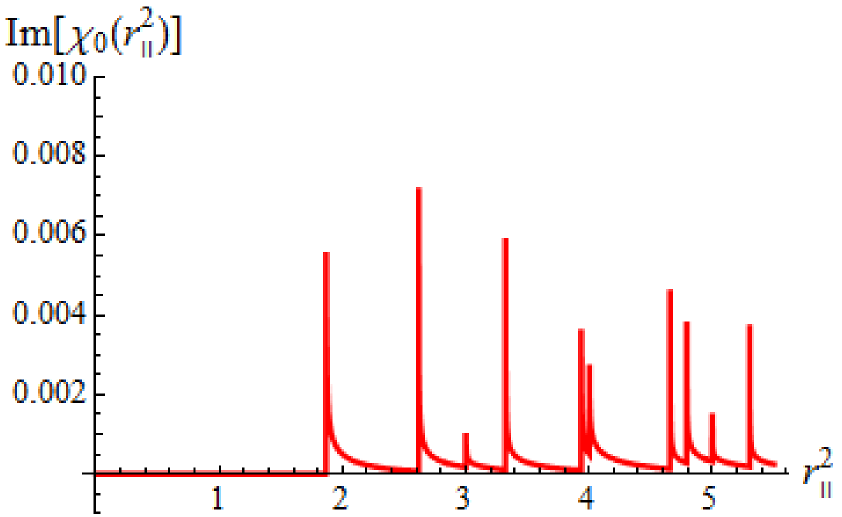
$$\chi_i = \frac{\alpha B_{\mathrm{r}}}{4\pi} e^{-\eta} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} \Omega_{\ell i}^{n}$$

$\Omega_{\ell i}^{n}$ is given by the Laguerre polynomials.
(Wave functions of charged fermions in B-fields)

Polarization tensor acquires an imaginary part above

$$q_\parallel^2 = \left[\sqrt{m^2 + 2\ell eB} + \sqrt{m^2 + 2(\ell+n)eB}\,\right]^2$$

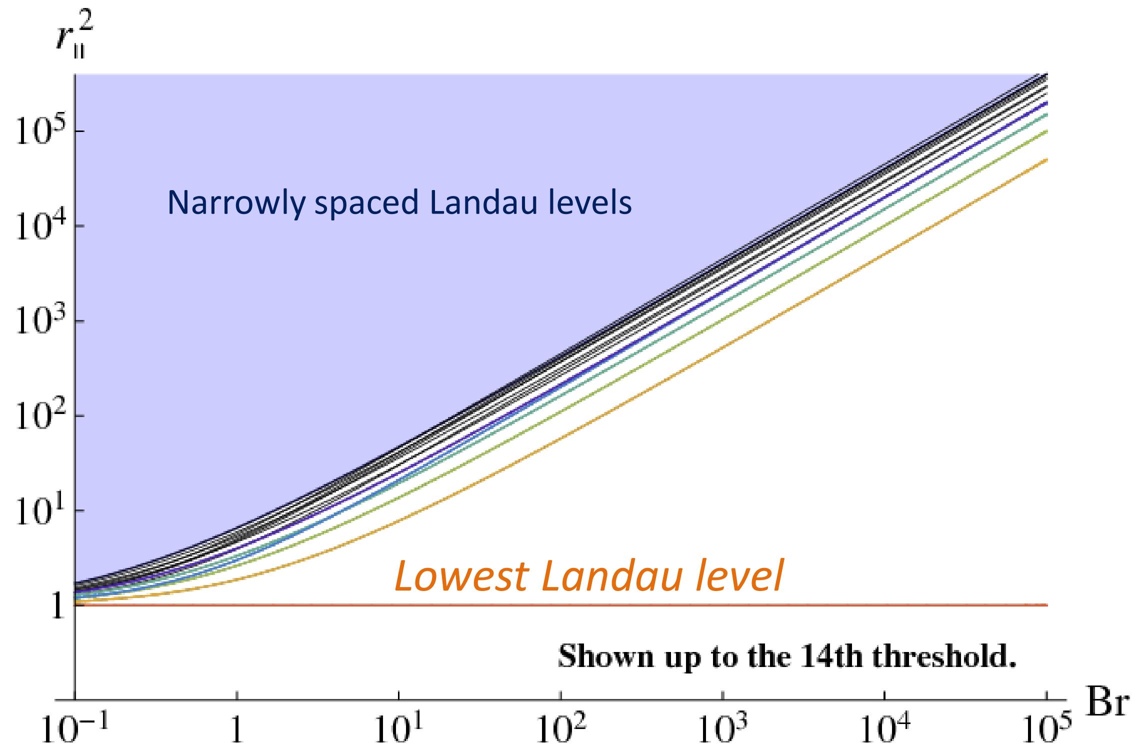
ℓ and n: "Landau levels" of a pair exchitation



Summary of relevant scales
- An infinite number of the Landau levels

$$r_\parallel^2 = \frac{q_\parallel^2}{4m_e^2}$$

(Photon momentum)



The first threshold
(lowest Landau level):
$q_\parallel^2 = 4m_e^2 = 1\ \mathrm{MeV}^2$

$$B_r = B/B_c$$

Complex refractive indices

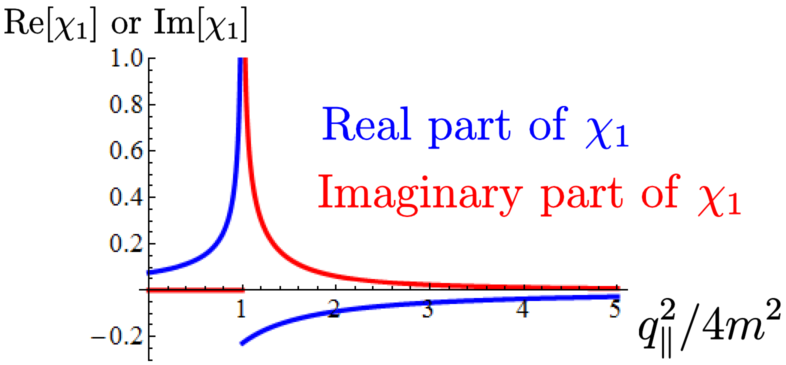
$$n = \frac{|\boldsymbol{q}|}{\omega}$$

Solutions of Maxwell eq.
with the vacuum polarization tensor

$$n_{\parallel}^2 = \frac{1+\chi_0+\chi_1}{1+\chi_0+\chi_1\cos^2\theta}$$
$$n_{\perp}^2 = \frac{1+\chi_0}{1+\chi_0+\chi_2\sin^2\theta}$$

The Lowest Landau Level (ℓ=n=0)

$$\chi_1 \neq 0, \quad \chi_0 = \chi_2 = 0$$



Refractive indices at the LLL

$$\begin{cases} n_{\parallel}^2 = \frac{1+\chi_1}{1+\chi_1\cos^2\theta} \\ n_{\perp}^2 = 1 \end{cases}$$

Polarization excites only along the magnetic field
``Vacuum birefringence''

Self-consistent solutions of the modified Maxwell Eq.

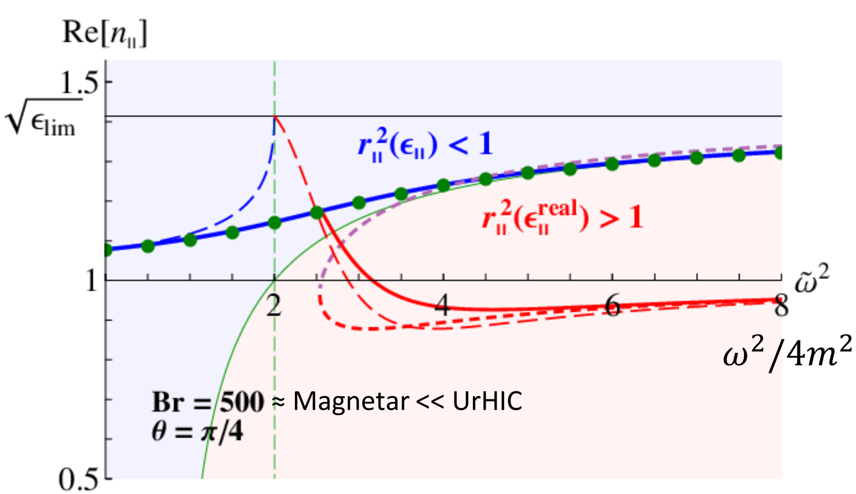
KH, K. Itakura (II)

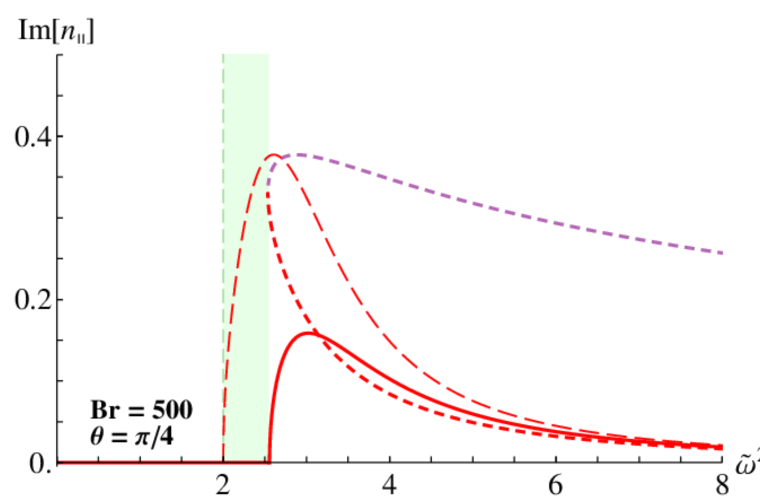
$$n^2 = \frac{1+\chi_1}{1+\chi_1 \cos^2\theta}$$

$\chi_1(q_\parallel^2, q_\perp^2; B_r)$

$$q_\parallel^2 = \omega^2 - q_z^2 = \omega^2(1 - n^2 \cos^2\theta)$$
$$q_\perp^2 = -|\boldsymbol{q}_\perp|^2 = -n^2 \omega^2 \sin^2\theta$$

Photon dispersion relation is strongly modified
when strongly coupled to excitations (cf: exciton-polariton, etc)



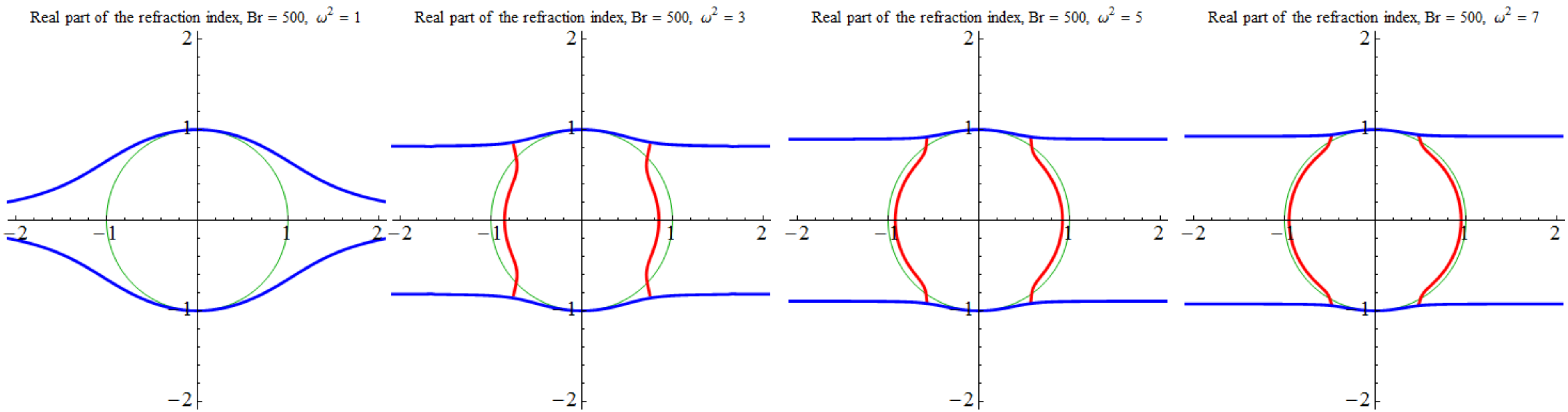


cf: air n = 1.0003, water n = 1.333

Angle dependence of the refractive index

$B_\mathrm{r} = 500$

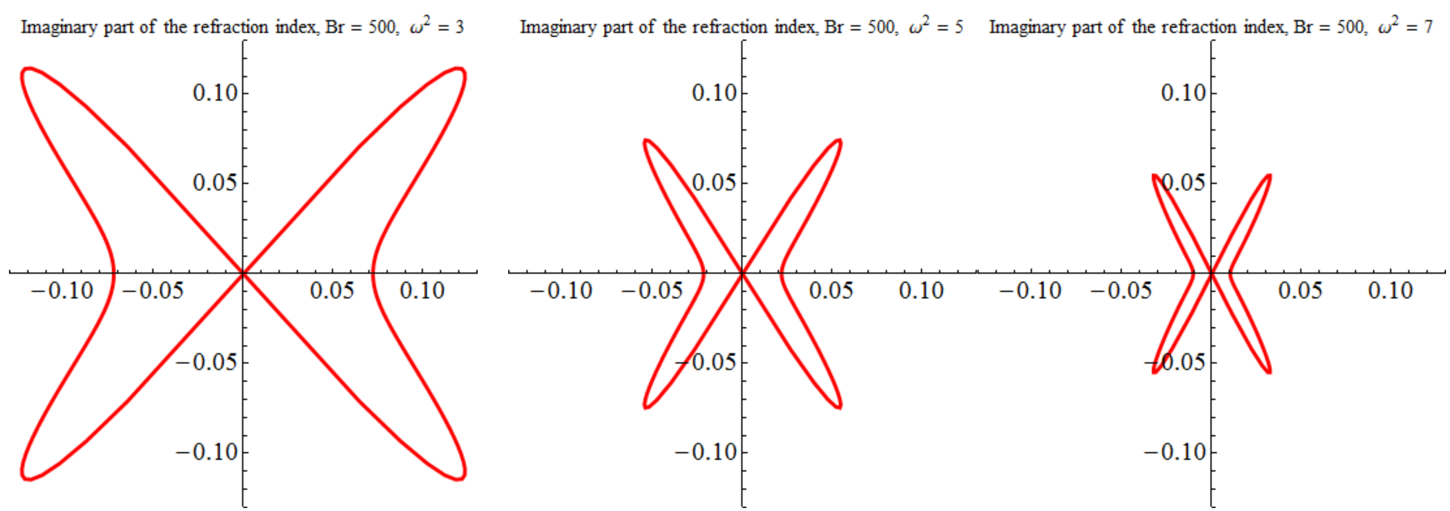
Real part



$\omega^2 = 1$ $\omega^2 = 3$ $\omega^2 = 5$ $\omega^2 = 7$

Imaginary part

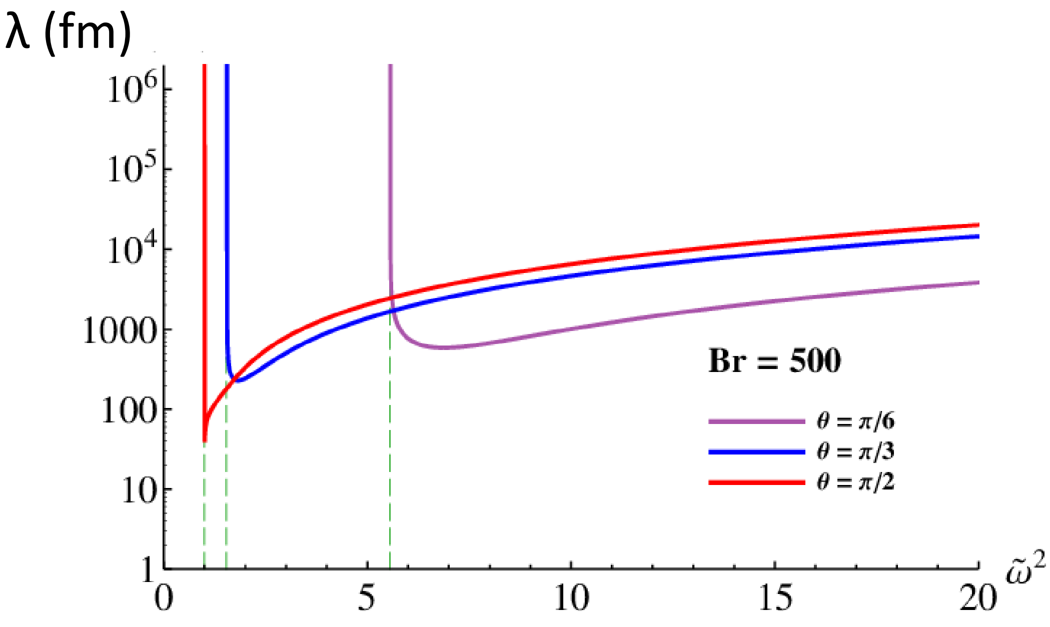
No imaginary part



“Mean-free-path” of photons in B-fields

$$\text{Photon flux}: \ \ I \propto \exp\{-\lambda^{-1}\hat{\boldsymbol{q}}\cdot\boldsymbol{x}\}$$

$$\lambda = \frac{1}{2\omega n_{\text{imag}}}$$



Summary

+ We obtained an analytic form of the polarization tensor
 in magnetic fields as the summation w.r.t. the Landau levels.

+ We obtained the complex refractive indices (photon dispersions)
 by solving the modified Maxwell Eq. self-consistently.
→ Photons decay within the microscopic spatial scale.

Prospects

We will go into phenomenology
in neutron stars/magnetars and the heavy-ion collisions .

The first seminal work in “nonlinear QED”

“Consequences of Dirac’s Theory of the Positron”

W. Heisenberg and H. Euler in Leipzig1

22. December 1935

Abstract

According to Dirac’s theory of the positron, an electromagnetic field tends to create pairs of particles which leads to a change of Maxwell’s equations in the vacuum. These changes are calculated in the special case that no real electrons or positrons are present and the field varies little over a Compton wavelength. The resulting effective Lagrangian of the field reads:

$$\mathfrak{L}=\frac{1}{2}(\mathfrak{E}^2-\mathfrak{B}^2)+\frac{e^2}{\hbar c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{conj.}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{conj.}} + |\mathfrak{E}_k|^2+\frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}$$

$\mathfrak{E}, \mathfrak{B}$ field strengths

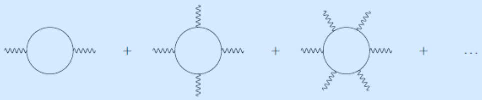
$|\mathfrak{E}_k|=\frac{m^2c^3}{e\hbar}=\frac{1}{137}\frac{e}{(e^2/mc^2)^2}=$ critical field strengths

The expansion terms in small fields (compared to $\mathfrak{E}$) describe light-light scattering. The simplest term is already known from perturbation theory. For large fields, the equations derived here differ strongly from Maxwell’s equations. Our equations will be compared to those proposed by Born.

Euler – Heisenberg effective Lagrangian

- resummation wrt the number of external legs

$$iS_{1-\text{loop}} = \ln\left[\det(i\not{D} - m)\right] =$$



Correct manipulation of a UV divergence in 1935!

Pair creation (vacuum instability) induced by strong electric field as known as Schwinger mechanism

Folgerungen aus der Diracschen Theorie des Positrons.

Von W. Heisenberg und H. Euler in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

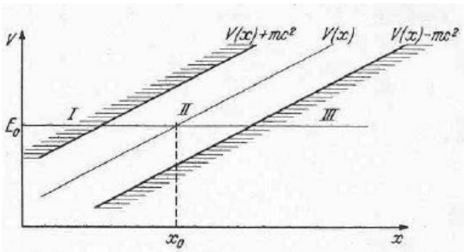


Fig. 1.

In Fig. 1, the potential energy $V(x)$ and the lines $V(x)+mc^2$ and $V(x)-mc^2$ are plotted against the coordinate (the electric field is parallel x-axis). The calculations of Sauter show that the eigenfunction associated to the eigenvalue E_0, for example, is large only in the regions I and II. In the region II, they decrease exponentially. Therefore, a wave function that begins being large in region I decreases slowly in region III where the transmission coefficient through region II (which plays the role of a Gamow-wall) calculated by Sauter has the order of magnitude $e^{-\frac{m^2c^3}{\hbar e|\mathfrak{E}|}\pi}$. If we define $|\mathfrak{E}_k| = \frac{m^2c^3}{\hbar e}$ as the critical field strength, we can also write $e^{-\frac{|\mathfrak{E}_k|}{|\mathfrak{E}|}\pi}$. As long as $|\mathfrak{E}| \ll |\mathfrak{E}_k|$, pair creation is so rare that it can be practically ignored. Then it must be possible to find

VOLUME 82, NUMBER 5

On Gauge Invariance and Vacuum Polarization

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts

(Received December 22, 1950)

If the integration path is considered to lie above the real axis, which is an alternative version of the device embodied in Eq. (6.32), we obtain a positive imaginary contribution to $\mathfrak{L}$,

$$2\,\mathrm{Im}\mathfrak{L} = \frac{1}{4\pi}\sum_{n=1}^{\infty} s_n^{-2}\exp(-m^2 s_n)$$

$$= \frac{\alpha^2}{\pi^2}\mathcal{E}^2 \sum_{n=1}^{\infty} n^{-2}\exp\left(\frac{-n\pi m^2}{e\mathcal{E}}\right). \quad (6.41)$$

This is the probability, per unit time and per unit volume, that a pair is created by the constant electric field.

General formula within 1-loop & constant field obtained by the "proper-time method".

Creation rate: $\Gamma \propto e^{-\frac{E_c}{E}\pi}$

Critical field: $E_c = \frac{m^2}{e}$

Quick derivation of the Landau level

Discretized spectrum of charged particle in the cyclotron orbits.

$$\text{Landau gauge}: \ \hat{\boldsymbol{A}} = (0, B\hat{x}_1, 0), \ \phi = 0$$

B = rot A = (0, 0, B)

$$\hat{H} = \frac{1}{2m^2}(\hat{\boldsymbol{p}} - e\hat{A})^2$$

$$= \frac{1}{2m^2}\left\{\hat{p}_1^2 + (\hat{p}_2 - eB\hat{x}_1)^2 + \hat{p}_3^2\right\}$$

$$[\hat{p}_2, \hat{H}] = 0 \text{ and } [\hat{p}_3, \hat{H}] = 0$$

$$\hat{H} = \frac{\hat{p}_1^2}{2m^2} + \frac{1}{2}m\omega_c^2(\hat{x}_1 - \frac{p_2}{eB})^2 + \frac{p_3^2}{2m^2}$$

$$\text{Cyclotron frequency}: \ \omega_c = \frac{eB}{m}$$

$$E_n = \omega_c\left(n + \tfrac{1}{2}\right) + \frac{p_3^2}{2m^2}, \quad (n \geq 0)$$

Harmonic oscillator + Continuous longitudinal spectrum

Wave function:
Associated Laguerre polynomial $L^{|n-m|}_{(n+m+|n-m|)/2}$

Relativistic version for spin-$\frac{1}{2}$:
$E_n = \sqrt{m^2 + 2eB\left(n + \frac{1}{2} \pm \frac{1}{2}\right) + p_3^2}$

It's convenient to use symmetric gauge,
$A = (-By, Bx, 0)/2$,
which respects the rotational symmetry.

What dynamics is encoded in the functions, χ_i $(i = 0, 1, 2)$?

$$\chi_i(r_\parallel^2, r_\perp^2, B_\mathrm{r}) = \frac{\alpha B_\mathrm{r}}{4\pi} \int_{-1}^{1} d\beta \int_0^\infty d\tau \, \frac{\Gamma_i(B_\mathrm{r}\tau, \beta)}{\sin(B_\mathrm{r}\tau)} \, e^{-i(\phi_\parallel + \phi_\perp)\tau}$$

Dimesionless variables

$$B_\mathrm{r} = \frac{B}{B_c} \qquad r_\parallel^2 = \frac{q_\parallel^2}{4m^2} \qquad r_\perp^2 = \frac{q_\perp^2}{4m^2}$$

Schwinger, Adler, Shabad, Urrutia, Tsai and Eber, Dittrich and Gies

$$\phi_\parallel(r_\parallel^2, B_\mathrm{r}) = 1 - (1-\beta^2)\, r_\parallel^2$$

$$\phi_\perp(r_\parallel^2, B_\mathrm{r}) = -2\, \frac{\cos(\beta B_\mathrm{r}\tau) - \cos(B_\mathrm{r}\tau)}{\sin(B_\mathrm{r}\tau)}\, r_\perp^2$$

Exponentiated trig-functions generate strongly oscillating behavior by arbitrarily high frequency.

$$\begin{cases} \Gamma_0(B_\mathrm{r}\tau, \beta) = \cos(\beta B_\mathrm{r}\tau) - \beta \sin(\beta B_\mathrm{r}\tau) \cot(B_\mathrm{r}\tau) \\ \Gamma_1(B_\mathrm{r}\tau, \beta) = (1-\beta^2)\cos(B_\mathrm{r}\tau) - \Gamma_0(B_\mathrm{r}\tau, \beta) \\ \Gamma_2(B_\mathrm{r}\tau, \beta) = 2\frac{\cos(\beta B_\mathrm{r}\tau) - \cos(B_\mathrm{r}\tau)}{\sin^2(B_\mathrm{r}\tau)} - \Gamma_0(B_\mathrm{r}\tau, \beta) \quad . \end{cases}$$

1$^{\mathrm{st}}$ step: Use a series expansion known as partial wave decomposition Baier and Katkov

$$e^{-iu\cos(\beta\tau)} = \sum_{n=0}^{\infty} (2 - \delta_{n0}) I_n(-iu)\; e^{in\beta\tau}$$

After 1st step:

$$\chi_i = \frac{\alpha}{4\pi}\int_{-1}^{1} d\beta \int_0^\infty d\tau \sum_{i=0}^{2} (2-\delta_{n0})\frac{\gamma_i^{(n)}(\tau,\beta)}{\sin\tau}\, e^{i\eta\cot\tau}\, e^{-i(\phi_\parallel - n\beta)\tau}$$

$$\begin{cases} \gamma_0^{(n)}(\tau,\beta) = \frac{1}{2}\{\, I_{n+1}(-iu) + I_{n-1}(-iu)\,\} - n\beta\,\eta^{-1} I_n(-iu)\cos\tau \\ \gamma_1^{(n)}(\tau,\beta) = (1-\beta^2) I_n(-iu)\cos\tau - \gamma_0^{(n)}(\tau,\beta) \\ \gamma_2^{(n)}(\tau,\beta) = \sin^{-2}\tau\,\{\, I_{n+1}(-iu) + I_{n-1}(-iu) - 2I_n(-iu)\cos\tau\,\} - \gamma_0^{(n)}(\tau,\beta) \end{cases}$$

2nd step:

$$e^{i\eta\cot\tau}\, I_n(-iu) = e^{-\eta} e^{-z\frac{2\eta}{1-z}}\, I_n\left(\frac{2(\eta^2 z)^{\frac{1}{2}}}{1-z}\right)$$

Put $z = \exp(-2i\tau)$ and use

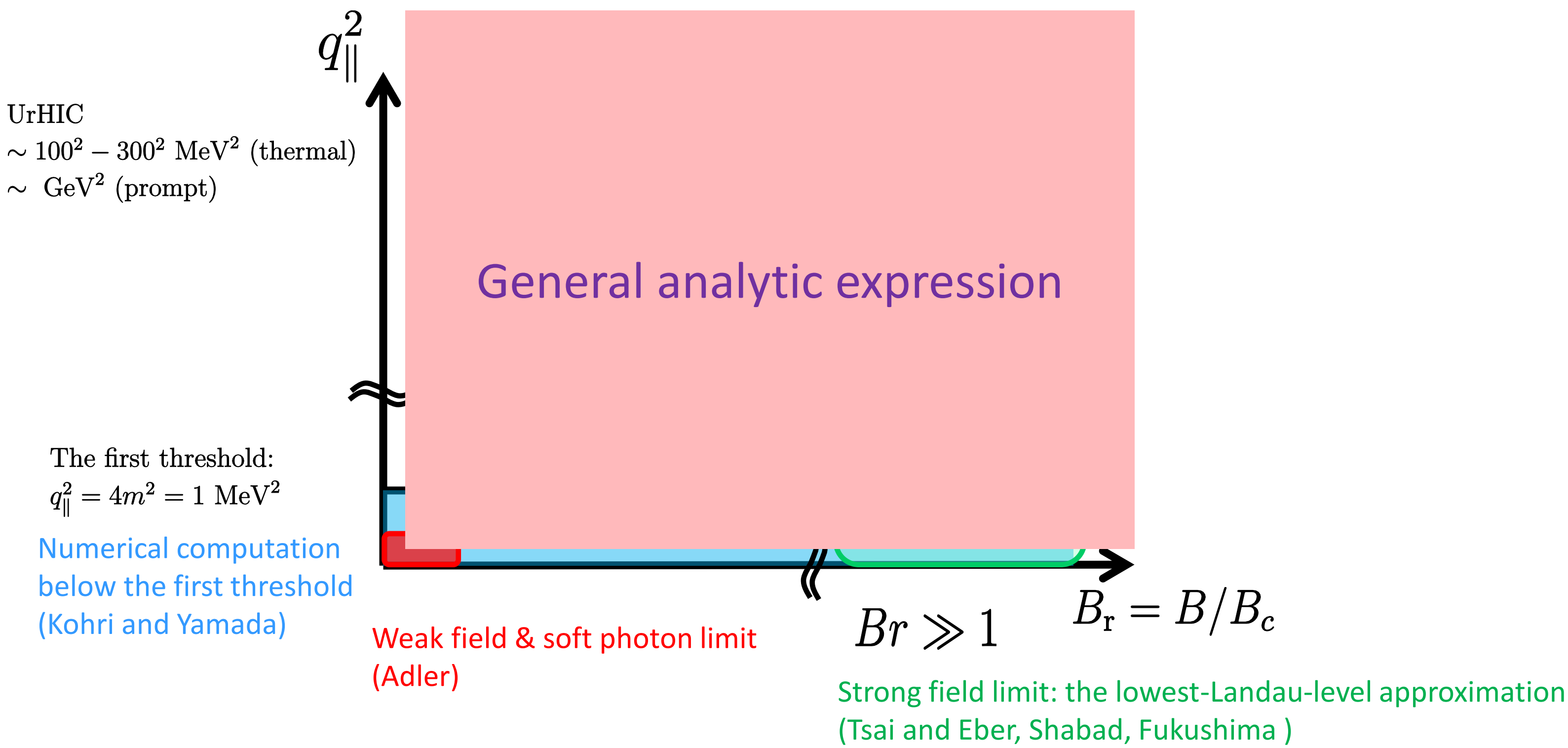
$$\exp\left(-a\frac{z}{1-z}\right) I_n\left(\frac{2az^{\frac{1}{2}}}{1-z}\right) = (1-z)\, a^n \sum_{\ell=0}^{\infty} \frac{\ell!}{\Gamma(\ell+n+1)}\, [L_\ell^n(a)]^2\, z^{\ell+\frac{n}{2}}$$

Associated Laguerre polynomial

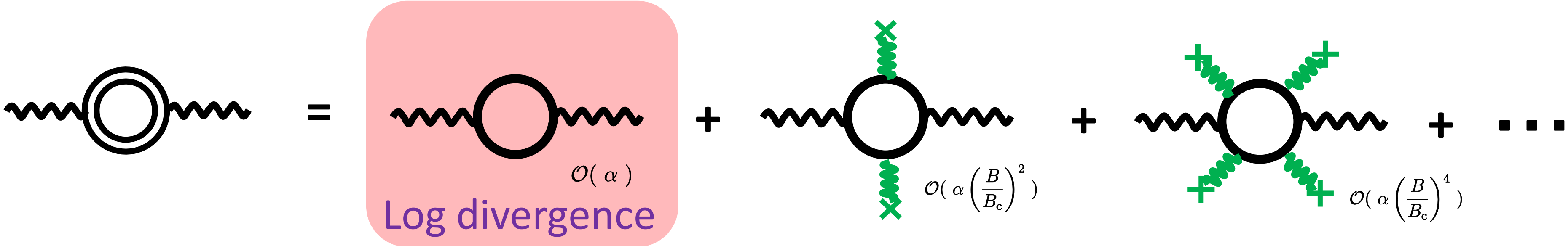
Then, any term reduces to either of three elementary integrals.

$$\begin{aligned} F_\ell^n(r_\parallel^2, B_r) &= \frac{i}{B_r}\int_{-1}^{1} d\beta \int_0^\infty d\tau\; e^{-i(\phi_\parallel + 2\ell - n\beta + n)\tau} \\ G_\ell^n(r_\parallel^2, B_r) &= \frac{i}{B_r}\int_{-1}^{1} d\beta \int_0^\infty d\tau\; \beta\, e^{-i(\phi_\parallel + 2\ell - n\beta + n)\tau} \\ H_\ell^n(r_\parallel^2, B_r) &= \frac{i}{B_r}\int_{-1}^{1} d\beta \int_0^\infty d\tau\; \beta^2\, e^{-i(\phi_\parallel + 2\ell - n\beta + n)\tau} \end{aligned}$$

Summary of relevant scales and preceding calculations



Renormalization

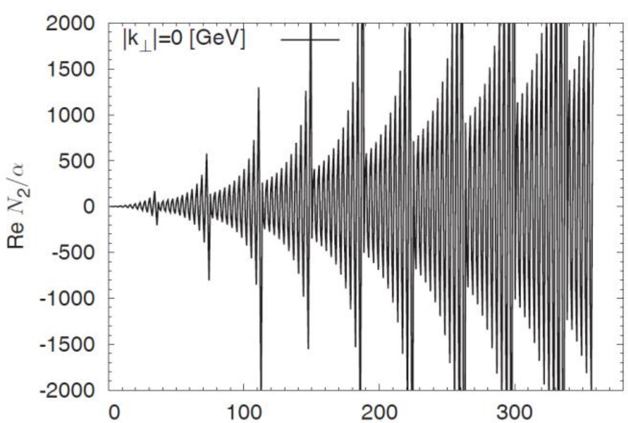


$$\begin{aligned}\Pi_{\text{ren}}(q^2) &= \Pi(q_\parallel^2, q_\perp^2) - \Pi_{\text{vac}}(q^2=0) \\ &= \underbrace{\Pi(q_\parallel^2, q_\perp^2) - \Pi(0, q_\perp^2)}_{\text{Subtraction term-by-term}} + \{\underbrace{\Pi(0, q_\perp^2) - \Pi_{\text{vac}}(q^2=0)}_{\text{Finite}}\}\end{aligned}$$

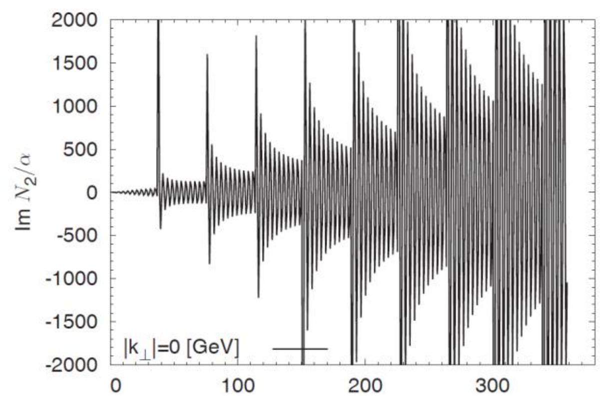
$\Pi(0, q_\perp^2)$ can be evaluated both by directly integrating the proper-time integrals and decomposing into the seires of Landau levels.

Ishikawa, Kimura, Shigaki, Tsuji (2013)

Re



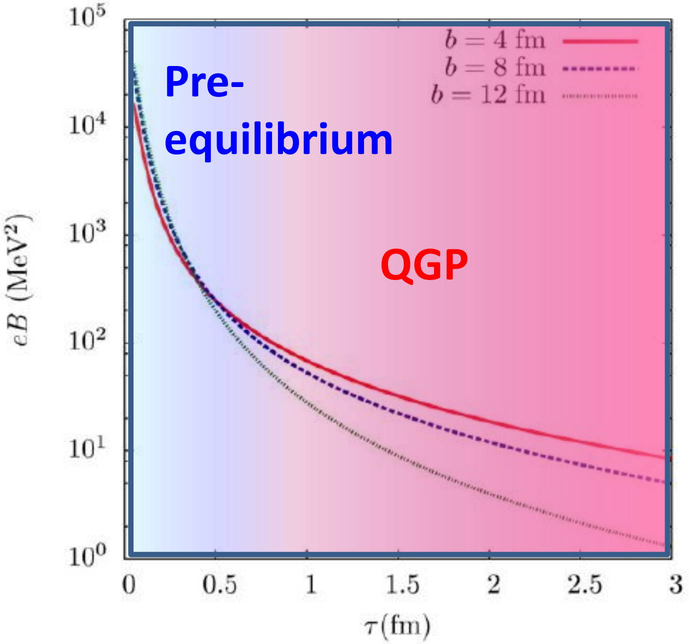
Im



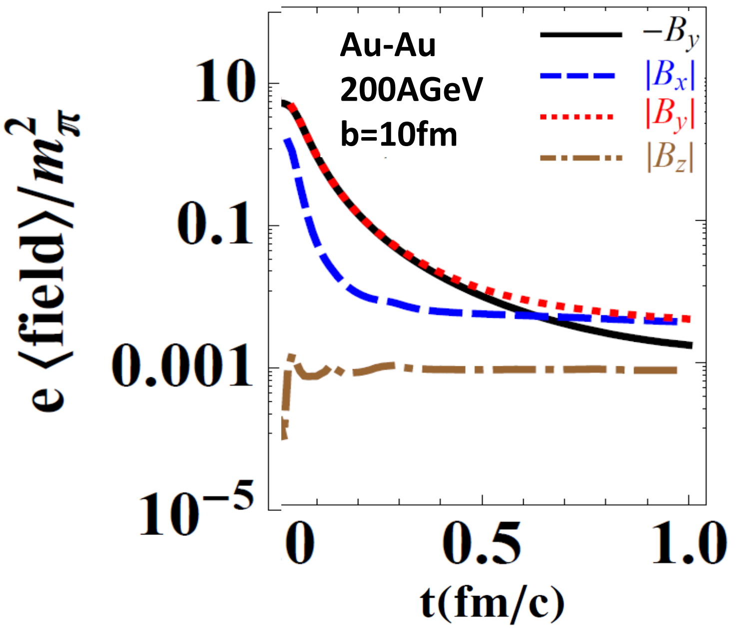
Taken from Ishikawa, et al. (2013)

Strong magnetic fields in UrHIC

Analytical modeling of colliding nuclei,
Kharzeev, McLerran, Warringa, NPA (2008)



Event-by-event analysis, Deng, Huang (2012)



Close look at the integrals

What dynamics is encoded in the scalar functions ?

$$\begin{aligned} F_\ell^n(r_\parallel^2, B_\mathrm{r}) &= I_{\ell\Delta}^n(r_\parallel^2) \\ G_\ell^n(r_\parallel^2, B_\mathrm{r}) &= \mathcal{G}_\ell^n[\ I_{\ell\Delta}^n(r_\parallel^2)\ ;\ r_\parallel^2, B_\mathrm{r}\ ] \\ H_\ell^n(r_\parallel^2, B_\mathrm{r}) &= \mathcal{H}_\ell^n[\ I_{\ell\Delta}^n(r_\parallel^2)\ ;\ r_\parallel^2, B_\mathrm{r}\ ] \end{aligned}$$

$$I_{\ell\Delta}^n(r_\parallel^2) = \frac{2}{\sqrt{4ac-b^2}}\left[\arctan\left(\frac{b+2a}{\sqrt{4ac-b^2}}\right) - \arctan\left(\frac{b-2a}{\sqrt{4ac-b^2}}\right)\right]$$

$$a = r_\parallel^2\ ,\quad b = -nB_\mathrm{r}\ ,\quad c = (1-r_\parallel^2) + (2\ell+n)B_\mathrm{r}$$

An imaginary part representing a real photon decay

$$b^2 - 4ac = 0 \quad \Leftrightarrow \quad (-nB_\mathrm{r})^2 - 4r_\parallel^2\left[(1-r_\parallel^2) + (2\ell+n)B_\mathrm{r}\right] = 0$$

$$\Leftrightarrow \quad q_\parallel^2 = \left[\sqrt{m^2+2\ell eB} + \sqrt{m^2+2(\ell+n)eB}\ \right]^2$$

Invariant mass of a fermion-pair in the Landau levels

Analytic representation of $\Pi^{\mu\nu}(q,B)$

$$\chi_i = \frac{\alpha B_\mathrm{r}}{4\pi} \sum_{n=0}^{\infty} (2-\delta_{n0}) \left[\sum_{\ell=0}^{\infty} \Omega_{\ell\, i}^{n(0)} + \sum_{\ell=1}^{\infty} \Omega_{\ell\, i}^{n(1)} + \sum_{\ell=2}^{\infty} \Omega_{\ell\, i}^{n(2)} \right] ,$$

$$\begin{cases}
\Omega_{\ell 0}^{n(0)} = (1-\delta_{n0})\, C_\ell^{n-1}(\eta)\, F_\ell^n(\xi, B_\mathrm{r}) - n\eta^{-1}\, C_\ell^n(\eta)\, G_\ell^n(\xi, B_\mathrm{r}) , \\
\Omega_{\ell 0}^{n(1)} = (1+\delta_{n0})\, C_{\ell-1}^{n+1}(\eta)\, F_\ell^n(\xi, B_\mathrm{r}) - n\eta^{-1}\, C_{\ell-1}^n(\eta)\, G_\ell^n(\xi, B_\mathrm{r}) , \\
\Omega_{\ell 0}^{n(2)} = 0 . \\
\Omega_{\ell 1}^{n(0)} = C_\ell^n(\eta) \left\{ F_\ell^n(\xi, B_\mathrm{r}) - H_\ell^n(\xi, B_\mathrm{r}) \right\} - \Omega_{\ell 0}^{n(0)} , \\
\Omega_{\ell 1}^{n(1)} = C_{\ell-1}^n(\eta) \left\{ F_\ell^n(\xi, B_\mathrm{r}) - H_\ell^n(\xi, B_\mathrm{r}) \right\} - \Omega_{\ell 0}^{n(1)} , \\
\Omega_{\ell 1}^{n(2)} = 0 , \\
\Omega_{\ell 2}^{n(0)} = -\Omega_{\ell 0}^{n(0)} , \\
\Omega_{\ell 2}^{n(1)} = D_\ell^{n(1)}(\eta)\, F_\ell^n(\xi, B_\mathrm{r}) - \Omega_{\ell 0}^{n(1)} , \\
\Omega_{\ell 2}^{n(2)} = D_\ell^{n(2)}(\eta)\, F_\ell^n(\xi, B_\mathrm{r}) .
\end{cases}$$

$$C_\ell^n(\eta) \equiv e^{-\eta} \frac{\ell!}{(\ell+n)!}\, \eta^n \left[L_\ell^n(\eta) \right]^2 .$$

$$D_\ell^{n(1)}(\eta) = -8 \sum_{\lambda=0}^{\ell-1} (\ell-\lambda) \left\{ (1-\delta_{n0})\, C_\lambda^{n-1}(\eta) - C_\lambda^n(\eta) \right\} ,$$

$$D_\ell^{n(2)}(\eta) = -8 \sum_{\lambda=0}^{\ell-2} (\ell-\lambda-1) \left\{ (1+\delta_{n0})\, C_\lambda^{n+1}(\eta) - C_\lambda^n(\eta) \right\} .$$

$$F_\ell^n(r_\parallel^2, B_\mathrm{r}) = \int_{-1}^{1} \frac{d\beta}{r_\parallel^2 \beta^2 - nB_\mathrm{r}\beta + (1-r_\parallel^2) + (2\ell+n)B_\mathrm{r}} \equiv I_{\ell\Delta}^n(r_\parallel^2)$$

$$G_\ell^n(r_\parallel^2, B_\mathrm{r}) = \frac{1}{2r_\parallel^2} \left[\Xi_\ell^n(B_\mathrm{r}) + nB_\mathrm{r}\, I_{\ell\Delta}^n(r_\parallel^2) \right] ,$$

$$H_\ell^n(r_\parallel^2, B_\mathrm{r}) = \frac{1}{r_\parallel^2} \left[2 + \frac{nB_\mathrm{r}}{2r_\parallel^2}\, \Xi_\ell^n(B_\mathrm{r}) + \frac{1}{4r_\parallel^2} \left\{ (b^2-4ac) + (nB_\mathrm{r})^2 \right\} I_{\ell\Delta}^n(r_\parallel^2) \right] ,$$

$$\Xi_\ell^n(B_\mathrm{r}) \equiv \ln \left| \frac{1+2\ell B_\mathrm{r}}{1+2(\ell+n)B_\mathrm{r}} \right| = \ln \left| \frac{m^2+2\ell eB}{m^2+2(\ell+n)eB} \right|$$

$$I_{\ell\Delta}^n(r_\parallel^2) = \begin{cases}
\frac{1}{\sqrt{(r_\parallel^2 - s_-^{\ell n})(r_\parallel^2 - s_+^{\ell n})}} \cdot \frac{1}{2} \ln \left| \frac{a-c-\sqrt{b^2-4ac}}{a-c+\sqrt{b^2-4ac}} \right| & (r_\parallel^2 < s_-^{\ell n}) \\
\frac{1}{\sqrt{\left|(r_\parallel^2 - s_-^{\ell n})(r_\parallel^2 - s_+^{\ell n})\right|}} \left[\arctan\left(\frac{b+2a}{\sqrt{4ac-b^2}} \right) - \arctan\left(\frac{b-2a}{\sqrt{4ac-b^2}} \right) \right] & (s_-^{\ell n} < r_\parallel^2 < s_+^{\ell n}) \\
\frac{1}{\sqrt{(r_\parallel^2 - s_-^{\ell n})(r_\parallel^2 - s_+^{\ell n})}} \cdot \frac{1}{2} \left[\ln \left| \frac{a-c-\sqrt{b^2-4ac}}{a-c+\sqrt{b^2-4ac}} \right| + 2\pi i \right] & (s_+^{\ell n} < r_\parallel^2) .
\end{cases}$$

- Infinite summation w.r.t. n and l = summation over two Landau levels
- Numerically confirmed by Ishikawa, et al. arXiv:1304.3655 [hep-ph]
- couldn’t find the same results starting from propagators with Landau level decomposition

Dielectric constant at the lowest-Landau-level

The first term $(\ell, n) = (0,0)$ in the double infinite series :

$$\chi_0 = 0$$
$$\chi_1 = \frac{\alpha B_r}{4\pi}\, e^{-\frac{|\boldsymbol{q}_\perp|^2}{2|eB|}} \times \frac{1}{r_\parallel^2}\left\{ I_{0\Delta}^0(r_\parallel^2) - 2 \right\}$$
$$\chi_2 = 0$$

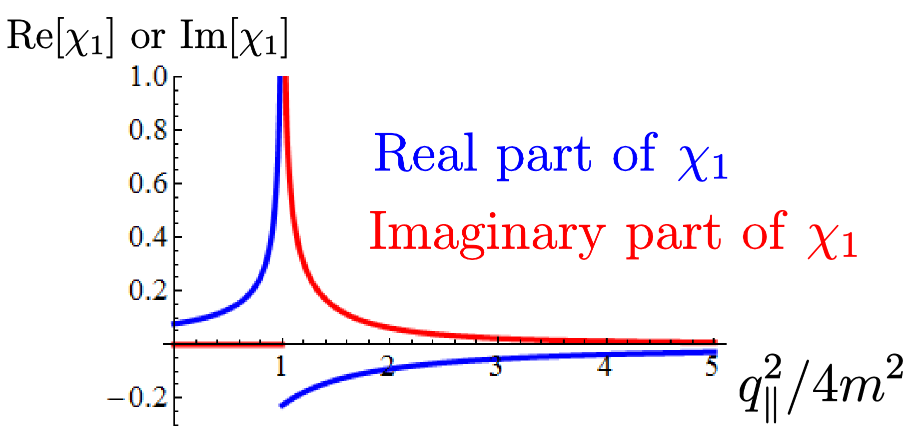
$$I_{0\Delta}^0(r_\parallel^2) = \frac{2}{\sqrt{r_\parallel^2(1-r_\parallel^2)}} \arctan\left(\frac{r_\parallel^2}{\sqrt{r_\parallel^2(1-r_\parallel^2)}}\right)$$

ArcTan : source of an imaginary part above the lowest threshold

Dielectric constant at the LLL

$$\begin{cases} \epsilon_\parallel = \frac{1+\chi_1}{1+\chi_1 \cos^2\theta} \\ \epsilon_\perp = 1 \end{cases}$$

Polarization excites only along the magnetic field



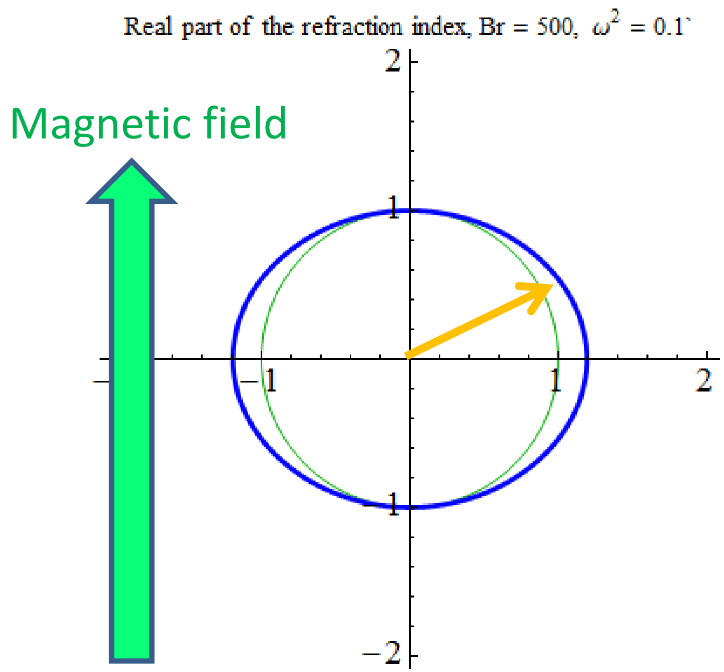
Complex refractive index

$$\begin{aligned} \epsilon &= \epsilon_r + i\epsilon_i \\ n &= n_r + i n_i \end{aligned}$$

$$n^2 = \epsilon \quad \Rightarrow \quad \begin{cases} n_r = \frac{1}{\sqrt{2}}\sqrt{|\epsilon| + \epsilon_r} \\ n_i = \frac{1}{\sqrt{2}}\sqrt{|\epsilon| - \epsilon_r} \end{cases} .$$

Angle dependence of the refractive index

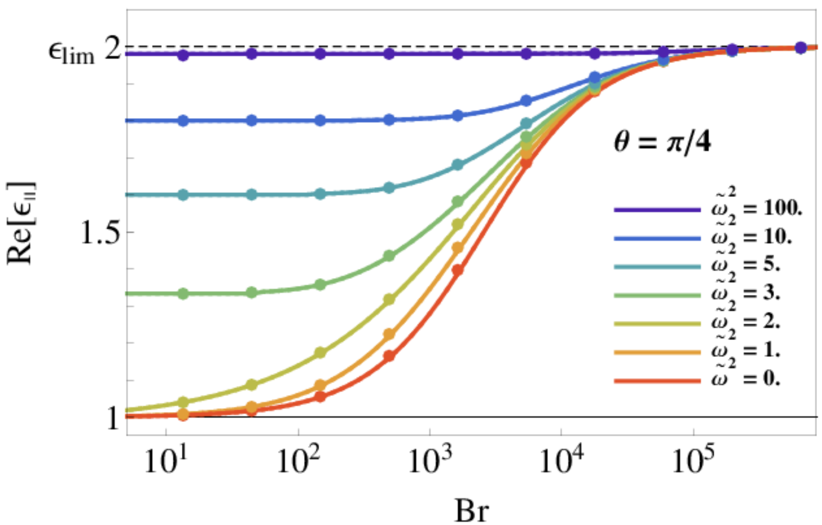
Shown as a deviation from unit circle



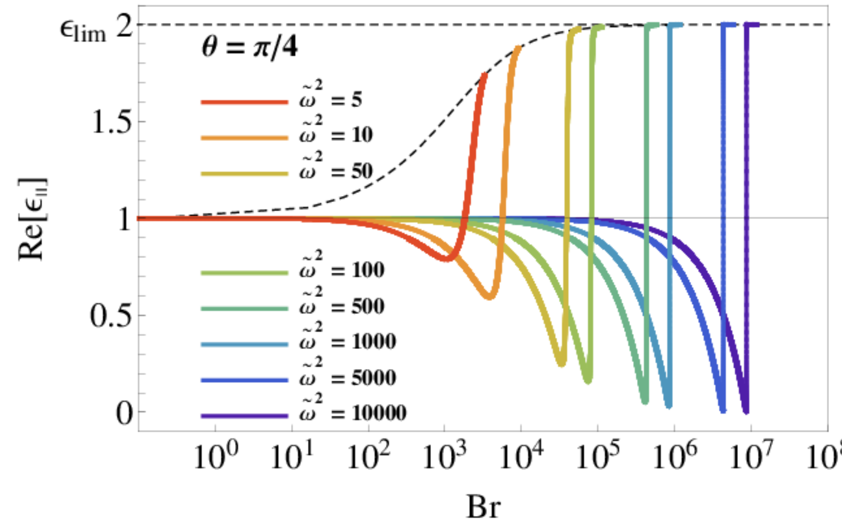
Direction of arrow :
direction of photon propagation

Norm of arrow :
magnitude of the refraction index

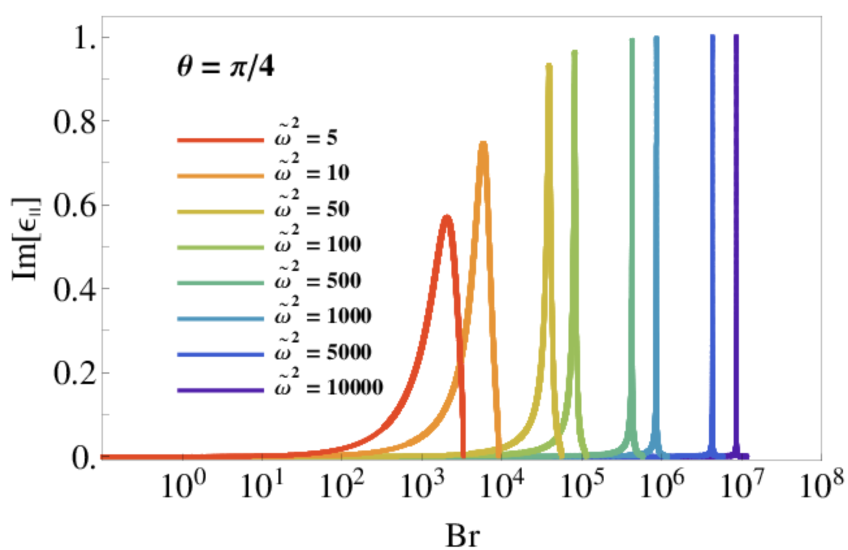
Real part of ε on stable branch



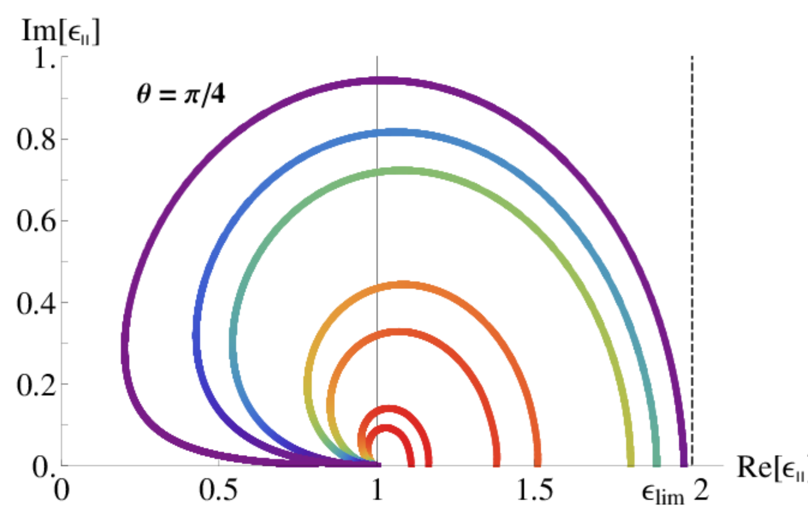
Real part of ε on unstable branch



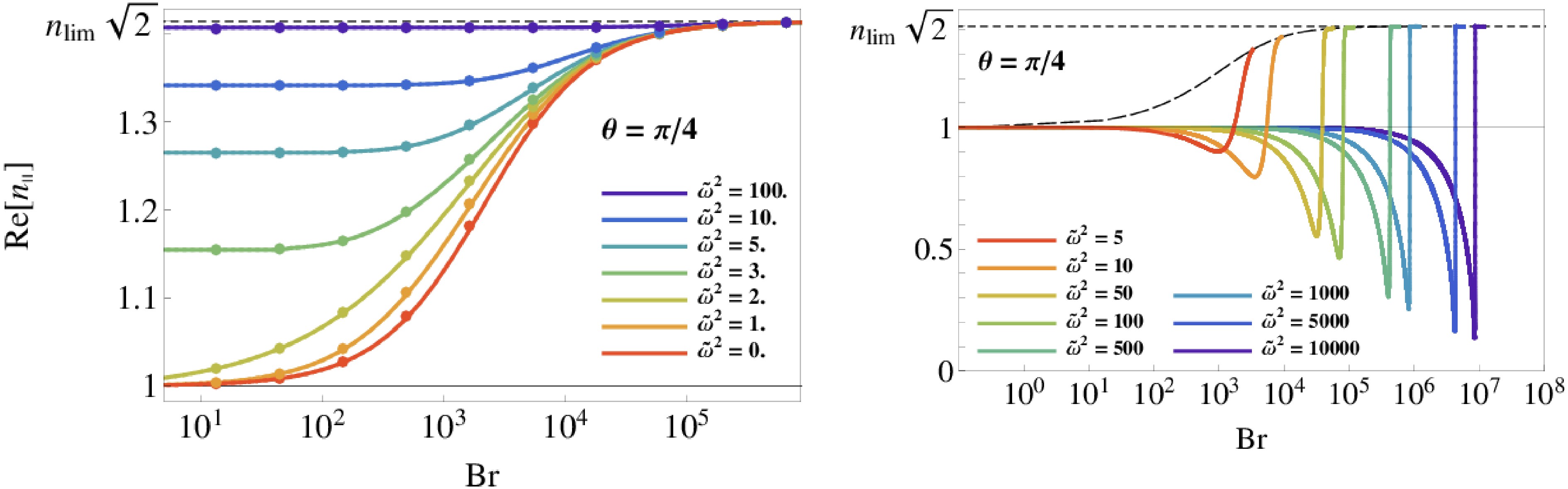
Imaginary part of ε on unstable branch



Relation btw real and imaginary parts on unstable branch



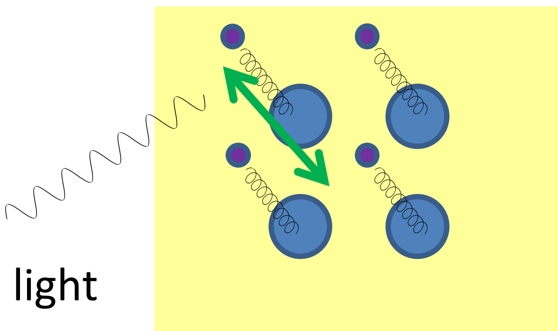
3r = (50,100,500,1000,5000,10000, 50000)



$n_{\lim}\sqrt{2}$
1.3
1.2
1.1
1
Re[$n_{\parallel}$]
$\theta = \pi/4$
$\tilde{\omega}^2$ = 100.
$\tilde{\omega}^2$ = 10.
$\tilde{\omega}^2$ = 5.
$\tilde{\omega}^2$ = 3.
$\tilde{\omega}^2$ = 2.
$\tilde{\omega}^2$ = 1.
$\tilde{\omega}^2$ = 0.
10^1 10^2 10^3 10^4 10^5
Br
$n_{\lim}\sqrt{2}$
$\theta = \pi/4$
1
0.5
0
$\tilde{\omega}^2$ = 5
$\tilde{\omega}^2$ = 10
$\tilde{\omega}^2$ = 50
$\tilde{\omega}^2$ = 100
$\tilde{\omega}^2$ = 500
$\tilde{\omega}^2$ = 1000
$\tilde{\omega}^2$ = 5000
$\tilde{\omega}^2$ = 10000
10^0 10^1 10^2 10^3 10^4 10^5 10^6 10^7 10^8
Br

Schematic picture of the birefringence

Polarization in dielectric medium :
a classical argument

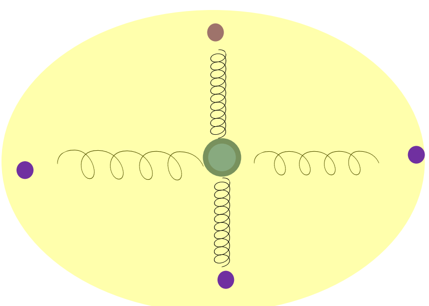


Lorentz-type dispersion :

$$\chi \propto \frac{1}{\omega^2 - \omega_0^2 + i\gamma\omega}$$

with characteristic frequency : $\omega_0^2 = U/m$

Anisotropic constants result in an anisotropic response.



$$\boldsymbol{P} = qN\boldsymbol{x} \quad \begin{cases} q : \text{charge} \\ N : \text{density of dipoles} \\ \boldsymbol{x} : \text{displacement} \end{cases}$$

What happens with the anisotropic (discretized) spectrum by the Landau-levels ?

$$m\ddot{\boldsymbol{x}} + \gamma\dot{\boldsymbol{x}} + U\boldsymbol{x} = q\boldsymbol{E}$$

Dissipation

Linear bound force

Incident light field