Neutron-proton dynamics and pion emission in heavy-ion collisions

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Theoretical studies by transport models for HIC (a part of D01)

Experimental data $\pi^-/\pi^+ = ?? \Rightarrow E_{\text{sym}}(2\rho_0) = ??$

- Pion production based on AMD+JAM calculation
  collaboration with N. Ikeno, Y. Nara and A. Ohnishi
- Clusters ($d$, $t$, $^3\text{He}$, $\alpha$) in heavy-ion collisions and in AMD
- Recent progress in the comparison of many transport codes
Various densities in heavy-ion collisions

\[ (E/A)(\rho_p, \rho_n) = (E/A)_0(\rho) + S(\rho)\delta^2 + \cdots \]

\[ \rho = \rho_p + \rho_n, \quad \delta = (\rho_n - \rho_p)/(\rho_n + \rho_p). \]

Constrains on \( S(\rho) \) at ICNT2013 @FRIB

Various densities in heavy-ion collisions

\[ \text{\^{132}Sn + \^{124}Sn, 300 MeV/u, } b \sim 0 \]

The S_πRIT project: TPC in SAMURAI magnet

- Pions are measured to probe high-density nuclear matter by the S_πRIT experiment, as proposed by Bao-An Li, PRL88 (2002) 192701.
- Many protons are bound in clusters.
  - In Au+Au at 250 MeV/u (FOPI data), p: 21%, \(\alpha\): 20%, d+t+^3\text{He}: 40%
Pion box simulation

A box simulation with a cascade code (JAM).

\[ \text{Number of particles} = \frac{p \times 1}{100} \times 100, \quad n \times 1 \times 100 \]

\[ \text{time [fm/c]} = 0, 100, 200 \]

\[ \text{DaPaEa} \quad \text{N768Z512BT060} \]

\[ N N \leftrightarrow N \Delta \quad \Delta \leftrightarrow N \pi \]

\[ \Delta^- \quad \Delta^0 \quad \Delta^+ \quad \pi^- \quad \pi^0 \quad \pi^+ \]

\[ \Rightarrow \text{Thermal and chemical equilibrium} \]

Initial condition:
- \( T = 60 \text{ MeV} \) (Relativistic Boltzmann)
- \( N = 768, \ Z = 512 \)
- \( V = (20 \text{ fm})^3 \) \( (\rho = 0.16 \text{ fm}^{-3}) \)
A heavy-ion collision is a highly dynamical process. It is not as simple as the systems in thermal and chemical equilibrium.

\[ 132_{\text{Sn}} + 124_{\text{Sn}}, \ E/A = 300 \text{ MeV, } b \sim 0, \ \text{AMD calculation} \]

**Question**

Is the relation \((N/Z)^2 = \pi^-/\pi^+\) still valid in HIC?

\[ \text{Symmetry Energy } S(\rho) \iff N/Z \iff \Delta^-/\Delta^0/\Delta^+/\Delta^{++} \iff \pi^-/\pi^+ \]

⇒ We need transport model calculations.
Transport equations for $N$, $\Delta$ and $\pi$

### Coupled equations for $f_N(r, p, t)$, $f_\Delta(r, p, t)$, $f_\pi(r, p, t)$

\[
\frac{\partial f_N}{\partial t} + \frac{\partial h_N}{\partial p} \cdot \frac{\partial f_N}{\partial r} - \frac{\partial h_N[f_N, f_\Delta, f_\pi]}{\partial r} \cdot \frac{\partial f_N}{\partial p} = I_N[f_N, f_\Delta, f_\pi] \quad \text{NN} \rightarrow \text{NN}
\]

\[
\frac{\partial f_\Delta}{\partial t} + \frac{\partial h_\Delta}{\partial p} \cdot \frac{\partial f_\Delta}{\partial r} - \frac{\partial h_\Delta[f_N, f_\Delta, f_\pi]}{\partial r} \cdot \frac{\partial f_\Delta}{\partial p} = I_\Delta[f_N, f_\Delta, f_\pi] \quad \text{NN} \leftrightarrow \text{N}\Delta
\]

\[
\frac{\partial f_\pi}{\partial t} + \frac{\partial h_\pi}{\partial p} \cdot \frac{\partial f_\pi}{\partial r} - \frac{\partial h_\pi[f_N, f_\Delta, f_\pi]}{\partial r} \cdot \frac{\partial f_\pi}{\partial p} = I_\pi[f_N, f_\Delta, f_\pi] \quad \Delta \leftrightarrow \text{N}\pi
\]

Assumption: $\Delta$ and pion productions are rare (in low-energy collisions), so that they can be treated as perturbation.

\[
I_N[f_N, f_\Delta, f_\pi] = I_N^{el}[f_N, 0, 0] + \lambda I'_N[f_N, f_\Delta, f_\pi]
\]

\[
f_N = f_N^{(0)} + \lambda f_N^{(1)} + \cdots, \quad f_\Delta = O(\lambda), \quad f_\pi = O(\lambda)
\]

Ikeno, Ono, Nara, Ohnishi, PRC 93 (2016) 044612.
Zeroth order equation for $f_N$

\[
\frac{\partial f_N^{(0)}}{\partial t} + \frac{\partial h_N}{\partial p} \cdot \frac{\partial f_N^{(0)}}{\partial r} - \frac{\partial h_N[f_N^{(0)},0,0]}{\partial r} \cdot \frac{\partial f_N^{(0)}}{\partial p} = I_N^{el}[f_N^{(0)},0,0]
\]

First order equations for $f_\Delta$ and $f_\pi$

\[
\frac{\partial f_\Delta}{\partial t} + \frac{\partial h_\Delta}{\partial p} \cdot \frac{\partial f_\Delta}{\partial r} - \frac{\partial h_\Delta[f_N^{(0)},f_\Delta,f_\pi]}{\partial r} \cdot \frac{\partial f_\Delta}{\partial p} = I_\Delta[f_N^{(0)},f_\Delta,f_\pi]
\]

\[
\frac{\partial f_\pi}{\partial t} + \frac{\partial h_\pi}{\partial p} \cdot \frac{\partial f_\pi}{\partial r} - \frac{\partial h_\pi[f_N^{(0)},f_\Delta,f_\pi]}{\partial r} \cdot \frac{\partial f_\pi}{\partial p} = I_\pi[f_N^{(0)},f_\Delta,f_\pi]
\]

JAM: Jet AA Microscopic transport model


Successful in high energy collisions (1 ~ 158 A GeV).
Antisymmetrized Molecular Dynamics (very basic version)


AMD wave function

\[ |\Phi(Z)\rangle = \det_{ij} \left[ \exp \left\{ -\nu \left( r_j - \frac{Z_i}{\sqrt{\nu}} \right)^2 \right\} \chi_{\alpha_i}(j) \right] \]

\[ Z_i = \sqrt{\nu} D_i + \frac{i}{2\hbar \sqrt{\nu}} K_i \]

\( \nu \): Width parameter = \((2.5 \text{ fm})^{-2} \)

\( \chi_{\alpha_i} \): Spin-isospin states = \( p \uparrow, p \downarrow, n \uparrow, n \downarrow \)

Equation of motion for the wave packet centroids \( Z \)

\[ \frac{d}{dt} Z_i = \{Z_i, \mathcal{H}\}_\text{PB} + (\text{NN collisions}) \]

\( \{Z_i, \mathcal{H}\}_\text{PB}: \) Motion in the mean field

\[ \mathcal{H} = \frac{\langle \Phi(Z)|H|\Phi(Z)\rangle}{\langle \Phi(Z)|\Phi(Z)\rangle} + \text{(c.m. correction)} \]

\( H \): Effective interaction (e.g. Skyrme force)

NN collisions

\[ W_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle \Psi_f | V | \Psi_i \rangle|^2 \delta(E_f - E_i) \]

- \(|V|^2 \) or \( \sigma_{NN} \) (in medium)
- Pauli blocking
NN collisions with cluster correlations

Even in collisions at several hundred collisions, **clusters are abundantly produced.** In Au+Au central collisions at 250 MeV/u (FOPI data, NPA848 (2010) 366),

\[
p: 21\%, \quad ^4\text{He}: 20\%, \quad d+t+^3\text{He}: 40\%
\]

In AMD, formation clusters is explicitly considered in the final state of each NN collision.

\[
N_1 + B_1 + N_2 + B_2 \rightarrow C_1 + C_2
\]

\[
W_{i\rightarrow f} = \frac{2\pi}{\hbar} |\langle CC|V_{NN}|NBNB\rangle|^2 \delta(E_f - E_i)
\]

\[
C_1, C_2 = n, p, d, \ldots, ^4\text{He}
\]


Ikeno, Ono et al., PRC 93 (2016) 044612

- The important ingredient is still the NN cross sections (or matrix elements $|V_{NN}|^2$).
- Processes like $d + X \rightarrow n + p + X'$ and $d + X \rightarrow d + X'$ are automatically considered.
- We also consider the process that several clusters form bound light nuclei. e.g. $\alpha + t \rightarrow ^7\text{Li} \ (\text{B.E.} = -2.5 \text{ MeV})$. 
Results for multifragmentation in central collisions

**Xe + Sn**


**Ca + Ca at 35 MeV/u**

**Au + Au at 250 MeV/u**
Nucleons and pions in HIC

A heavy-ion collision is a highly dynamical process. It is not be as simple as the systems in thermal and chemical equilibrium.

\[ ^{132}\text{Sn} + ^{124}\text{Sn}, \quad E/A = 300 \text{ MeV}, \quad b \sim 0, \quad \text{AMD calculation} \]

**Question**

Is the relation \((N/Z)^2 = \pi^-/\pi^+\) still valid in HIC?

\[ \text{Symmetry Energy } S(\rho) \iff N/Z \iff \Delta^-/\Delta^0/\Delta^+/\Delta^{++} \iff \pi^-/\pi^+ \]

\[ \Rightarrow \text{We need transport model calculations.} \]
Does $\Delta$ production agree with n/p dynamics

Ikeno, Ono, Nara, Ohnishi, PRC 93 (2016) 044612.

Different n/p dynamics $\iff$ different $E_{\text{sym}}(\rho)$ and cluster correlations

Four AMD+JAM calculations and a JAM calculation

- Solid lines: with clusters; **Soft** ($L = 46$ MeV) and **Stiff** ($L = 108$ MeV)
- Dashed lines: without clusters; **Soft** ($L = 46$ MeV) and **Stiff** ($L = 108$ MeV)
- Dot-dashed line: JAM (without mean field)
Does $\Delta$ production agree with n/p dynamics

Ikeno, Ono, Nara, Ohnishi, PRC 93 (2016) 044612.

$\left(\frac{N}{Z}\right)^2$ @ high density and high momentum

Rate$(nn \rightarrow p\Delta^-) / Rate(pp \rightarrow n\Delta^{++})$

(b) $\rho(r) \geq \rho_0$, $|p - p_{rad}| \geq 480$ MeV/c

Different n/p dynamics $\iff$ different $E_{sym}(\rho)$ and cluster correlations

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- Dashed lines: without clusters; **Soft** ($L = 46$ MeV) and **Stiff** ($L = 108$ MeV)
- Dot-dashed line: JAM (without mean field)
Summary of ratios, for $^{132}$Sn + $^{124}$Sn at 300 MeV/nucleon

- $(N/Z)^2$, $(N/Z)_{\rho,p}^2$, $\Delta^-/\Delta^{++}$: representative values integrated (or averaged) over time.
- $(N/Z)^2$ in high-$\rho$ and high-$p$ region $\approx (\pi^-/\pi^+)_\text{like}$ at $t = 20 \text{ fm/c}$ $\cdots$ Final $\pi^-/\pi^+$ $\approx 30\%$ reduction of $E_{\text{sym}}$ effect
- Effects of clusters: Larger ratios, and weaker dependence on $E_{\text{sym}}$
- A kind of model dependence (with or without clusters)

Ikeno, Ono, Nara, Ohnishi, PRC 93 (2016) 044612
Symmetry energy and pions studied by other transport models

**Figure by Hong et al.**

- Bao-An Li, PRL 88 (2002) 192701. IBUU
- Reisdorf et al., NPA 781 (2007) 459. Data & IQMD
- Z. Xiao et al., PRL 102 (2009) 062502. IBUU04
- Z.Q. Feng and G.M. Jin, PLB 683 (2010) 140. ImIQMD
- Hong and Danielewicz, PRC 90 (2014) 024605. pBUU
- ...

Model predictions do not agree.

⇒ What to do?

<table>
<thead>
<tr>
<th></th>
<th>Stiff $E_{sym}$</th>
<th>Soft $E_{sym}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IBUU</td>
<td>$\pi^-/\pi^+$</td>
<td>$\pi^-/\pi^+$</td>
</tr>
<tr>
<td>ImIQMD</td>
<td>$\pi^-/\pi^+$</td>
<td>$\pi^-/\pi^+$</td>
</tr>
<tr>
<td>pBUU</td>
<td>$\pi^-/\pi^+$</td>
<td>$\pi^-/\pi^+$</td>
</tr>
</tbody>
</table>
To improve predictions by transport models

It is now very important to reduce the theoretical ambiguities. How can we do?

- Use or develop a good transport model.
  AMD with clusters + JAM. But it depends on ...

- Find/understand mechanisms which do not depend much on models.
  e.g., Symmetry Energy \( \leftrightarrow (N/Z)^2 \rho, p \leftrightarrow \Delta^- / \Delta^{++} \leftrightarrow \pi^- / \pi^+ \)

- Compare results of transport models to find the sources of discrepancies.
  This requires much efforts by many people.
Understanding transport simulations of heavy-ion collisions at 100A and 400A MeV:
Comparison of heavy-ion transport codes under controlled conditions

J. Xu et al.,
PRC93 (2016)
044609.
Uncertainties in the flow parameter: about 30% (100 MeV), about 13% (400 MeV)
Next phase of Transport Code Comparison: Box simulations

We have found discrepancies in the number of NN collisions. What are the sources? We must disentangle many possibilities (initialization, NN collisions, Pauli blocking, mean field, ...)

⇒ Let’s do Box Simulations.

We still had large divergence. (≈ 30%)

⇒ Many codes fixed an issue of the Bertsch prescription for NN collisions.

⇒ Much better agreement! (a few %)

- Simple initialization \((T = 0\) Fermi)
- No mean field
- \(\sigma_{NN} = 40\) mb
- Turn off Pauli blocking

Next (1) Pauli blocking
Next (2) with mean field
Next (3) \(\pi\) and \(\Delta\) in the box
Transport calculations for HIC

Experimental data $\pi^-/\pi^+ = \cdots \Rightarrow E_{\text{sym}}(2\rho_0) = \cdots$

- Use or develop a good transport model.
  AMD with clusters + JAM.

- Find/understand mechanisms which do not depend much on models.
  Symmetry Energy $\leftrightarrow (N/Z)^2_{\rho,p} \leftrightarrow \Delta^-/\Delta^{++} \leftrightarrow \pi^-/\pi^+$

- Compare results of transport models to find the sources of discrepancies.
  Transport code comparison is going on.

\[ (N/Z)^2_{\rho}, \pi^-/\pi^+ \]

\[ (N/Z)^2_{\rho}, \Delta^-/\Delta^{++}, \pi^-/\pi^+ \]

\[ \text{ratio} \]

\[ \text{with cl., soft} \quad \text{with cl., stiff} \quad \text{w/o cl., soft} \quad \text{w/o cl., stiff} \quad \text{JAM} \]
$N/Z$ of spectrum of emitted particles is similar to the neutron-proton density difference at the compression stage.

\[
\left( \frac{N}{Z} \right)_{\text{gas}} = \frac{Y_n(v) + Y_d(v) + 2Y_t(v) + Y_h(v) + 2Y_{\alpha}(v)}{Y_p(v) + Y_d(v) + Y_t(v) + 2Y_h(v) + 2Y_{\alpha}(v)}
\]
The $N/Z$ spectrum ratio (AMD without clusters) is shown in the graph.

The equation for the $N/Z$ ratio of the gas is given by:

$$\left(\frac{N}{Z}\right)_{\text{gas}} = \frac{Y_n(v) + Y_d(v) + 2Y_t(v) + Y_h(v) + 2Y_\alpha(v)}{Y_p(v) + Y_d(v) + Y_t(v) + 2Y_h(v) + 2Y_\alpha(v)}$$

$N/Z$ of spectrum of emitted particles is NOT similar to the neutron-proton density difference at the compression stage.
JAM: Jet AA Microscopic transport model

- Applied to high-energy collisions (1 ~ 158A GeV)
- Hadron-Hadron reactions are based on experimental data and the detailed balance.
  
  \[
  \frac{d\sigma_{NN\rightarrow N\Delta}}{dm} = \frac{C_I}{p_is} \frac{|M|^2}{16\pi} \times \frac{2}{\pi} \frac{m^2\Gamma(m)}{(m^2 - m^2_\Delta)^2 + m^2\Gamma(m)^2} \left|f(m,s)\right|
  \]

  \[
  |M|^2 = \frac{s\Gamma^2_\Delta}{(s - m^2_\Delta)^2 + s\Gamma^2_\Delta}, \quad m_N + m_\pi < m < \sqrt{s} - m_N
  \]

- No mean field (default).
- s-wave pion production (NN → NNNπ) is turned off.
Pions from JAM and AMD+JAM calculations

Au + Au central collisions


The energy violation, due to the mismatch of AMD and JAM, is about 2 MeV/nucleon on average at \( t \approx 20 \text{ fm/c} \) in collisions at 300 MeV/nucleon. This corresponds to the 10% overestimation of the pion multiplicity.
Clustering phenomena in excited states of nuclear systems

$E^* \sim 8A$ MeV  Gas of clusters at higher energies

- Excitation energy / temperature
- Multifragmentation in heavy ion collision
- Molecular resonance
- Cluster decay
- Liquid-gas phase transition
- Collective modes (GR, PR)
- Weakly bound systems
- Shell evolution
- Deformation
- Shell structure cluster breaking
- Neutron-rich
- Halo, skin
- NN correlation
- Shell evolution
- Molecular orbitals
- Deformation
- Cluster breaking
- Shell structure
- Neutron-rich
- Halo, skin
- NN correlation
- Shell evolution
- Molecular orbitals
- Deformation
- Cluster breaking
- Shell structure

Interacting and reacting clusters in heavy-ion collisions

<table>
<thead>
<tr>
<th>Partitioning of protons</th>
<th>Xe + Sn 50 MeV/u</th>
<th>Au + Au 250 MeV/u</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>( \approx 10% )</td>
<td>21%</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( \approx 20% )</td>
<td>20%</td>
</tr>
<tr>
<td>d, t, ( ^3)He</td>
<td>( \approx 10% )</td>
<td>40%</td>
</tr>
<tr>
<td>( A &gt; 4 )</td>
<td>( \approx 60% )</td>
<td>18%</td>
</tr>
</tbody>
</table>

- The system may be composed of many clusters.
- Clusters are not only created but also broken by reactions.

Transport models with clusters + decay
Effect of cluster correlations: central Xe + Sn at 50 MeV/u

Without clusters

With clusters
HIC is a unique opportunity to study nuclear matter in laboratories:

- High and low densities: \( \sim 2 \rho_0 \rightarrow \frac{1}{2} \rho_0 \rightarrow \frac{1}{10} \rho_0 \rightarrow \)
- But it is just a small piece of nuclear matter with a few hundred nucleons.
- But it lives only for a short time (\( \sim 10 \text{ fm/c} \approx 3 \times 10^{-23} \text{ s} \)).

\[ ^{132}\text{Sn} + ^{124}\text{Sn}, \quad E/A = 300 \text{ MeV}, \quad b \sim 0 \]

AMD calculation

\[ \text{QCD} \rightarrow V_{NN} \rightarrow \text{EOS} \rightarrow \text{HIC} \]
HIC is a unique opportunity to study nuclear matter in laboratories:

- High and low densities: $\rightarrow \sim 2\rho_0 \rightarrow \frac{1}{2}\rho_0 \rightarrow \frac{1}{10}\rho_0 \rightarrow$
- But it is just a small piece of nuclear matter with a few hundred nucleons.
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$^{132}\text{Sn} + ^{124}\text{Sn}$, $E/A = 300 \text{ MeV}$, $b \sim 0$

AMD calculation

QCD $\rightarrow V_{NN}$

Effective interaction (Mean field)

NN collision cross sections

Transport theories

EOS $\rightarrow$ HIC
About potentials for $\Delta$ and $\pi$

<table>
<thead>
<tr>
<th>$N_{\tau_1} + N_{\tau_2}$</th>
<th>$N_{\tau_3} + \Delta_{\tau_4}$</th>
</tr>
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<tbody>
<tr>
<td>$U^{(N)}<em>{\tau_1} + U^{(N)}</em>{\tau_2}$</td>
<td>$U^{(N)}<em>{\tau_3} + U^{(\Delta)}</em>{\tau_4}$</td>
</tr>
<tr>
<td>$+ q_1 U_C + q_2 U_C$</td>
<td>$+ q_3 U_C + q_4 U_C$</td>
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<table>
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<tr>
<th>$\Delta_{\tau_1}$</th>
<th>$N_{\tau_3} + \pi_{\tau_4}$</th>
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<tr>
<td>$U^{(\Delta)}_{\tau_1}$</td>
<td>$U^{(N)}<em>{\tau_3} + U^{(\pi)}</em>{\tau_4}$</td>
</tr>
<tr>
<td>$+ q_1 U_C$</td>
<td>$+ q_3 U_C + q_4 U_C$</td>
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- $U^{(*)}_{\tau}$: Isospin($\tau$)-dependent potential due to the strong interaction
- $U_C$: Coulomb potential

In JAM, reaction thresholds are the same as in free space. Therefore AMD+JAM assumes

$$U^{(N)}_{\tau_1} + U^{(N)}_{\tau_2} = U^{(N)}_{\tau_3} + U^{(\Delta)}_{\tau_4}, \quad U^{(\Delta)}_{\tau_1} = U^{(N)}_{\tau_3} + U^{(\pi)}_{\tau_4}$$

for $\tau_1 (+\tau_2) = \tau_3 + \tau_4$

This is satisfied in case

$$U^{(N,\Delta)}_{\tau} = U_0(r) + \tau U_{\text{sym}}(r), \quad U^{(\pi)}_{\tau} = \tau U_{\text{sym}}(r)$$

## Difference choices of $\Delta$ potential

**Our choice**

- $V_{\text{asy}}(\Delta^-) = 3V_{\text{asy}}(n)$
- $V_{\text{asy}}(\Delta^0) = V_{\text{asy}}(n)$
- $V_{\text{asy}}(\Delta^+) = V_{\text{asy}}(p)$
- $V_{\text{asy}}(\Delta^{++}) = 3V_{\text{asy}}(p)$

**Another choice**

- $V_{\text{asy}}(\Delta^-) = V_{\text{asy}}(n)$
- $V_{\text{asy}}(\Delta^0) = \frac{1}{3}V_{\text{asy}}(n)$
- $V_{\text{asy}}(\Delta^+) = \frac{1}{3}V_{\text{asy}}(p)$
- $V_{\text{asy}}(\Delta^{++}) = V_{\text{asy}}(p)$

---


**Different choices of $\Delta$ potential**

$\Rightarrow$ Different thresholds for $\Delta$ production

$\Rightarrow$ Different $\pi^-/\pi^+$ ratios
Skyrme force

\[ v_{ij} = t_0 (1 + x_0 P_\sigma) \delta (r) + \frac{1}{2} t_1 (1 + x_1 P_\sigma) [\delta (r) \mathbf{k}^2 + \mathbf{k}^2 \delta (r)] + t_2 (1 + x_2 P_\sigma) \mathbf{k} \cdot \delta (r) \mathbf{k} + t_3 (1 + x_3 P_\sigma) [\rho (r_i)]^\alpha \delta (r) \]

\[ \mathbf{r} = r_i - r_j \]

\[ \mathbf{k} = \frac{1}{2\hbar} (\mathbf{p}_i - \mathbf{p}_j) \]

The expectation value can be written by using several kinds of densities.

\[ \langle V \rangle = \int V (\rho (r), \tau (r), \Delta \rho (r), j (r)) d\mathbf{r} \sim A^2 \times \text{Volume} \]

\[ \rho^\alpha (r) = \int f^\alpha (r, p) \frac{d\mathbf{p}}{(2\pi \hbar)^3} = \left( \frac{2\nu}{\pi} \right)^{\frac{3}{2}} \sum_{i \in \alpha} \sum_{j \in \alpha} e^{-2\nu (r - R_{ij})^2} B_{ij}^{-1} B_{ji}^{-1}, \quad \mathbf{R}_{ij} = \frac{1}{2\sqrt{\nu}} (Z_i^* + Z_j) \]

\[ j^\alpha (r) = \int \frac{\mathbf{p}}{M} f^\alpha (r, p) \frac{d\mathbf{p}}{(2\pi \hbar)^3} = \left( \frac{2\nu}{\pi} \right)^{\frac{3}{2}} \sum_{i \in \alpha} \sum_{j \in \alpha} \frac{\mathbf{P}_{ij}}{M} e^{-2\nu (r - R_{ij})^2} B_{ij}^{-1} B_{ji}^{-1}, \quad \mathbf{P}_{ij} = i\hbar \sqrt{\nu} (Z_i^* - Z_j) \]

\[ \tau^\alpha (r) = \int \frac{\mathbf{p}^2}{M^2} f^\alpha (r, p) \frac{d\mathbf{p}}{(2\pi \hbar)^3} = \left( \frac{2\nu}{\pi} \right)^{\frac{3}{2}} \sum_{i \in \alpha} \sum_{j \in \alpha} \frac{\mathbf{P}_{ij}^2 + 3\hbar^2 \nu}{M^2} e^{-2\nu (r - R_{ij})^2} B_{ij}^{-1} B_{ji}^{-1} \]

Momentum dependence of Skyrme (SLy4) interaction has been corrected for high energy collisions, in a similar way to Gale, Bertsch, Das Gupta, PRC 35 (1987) 1666.